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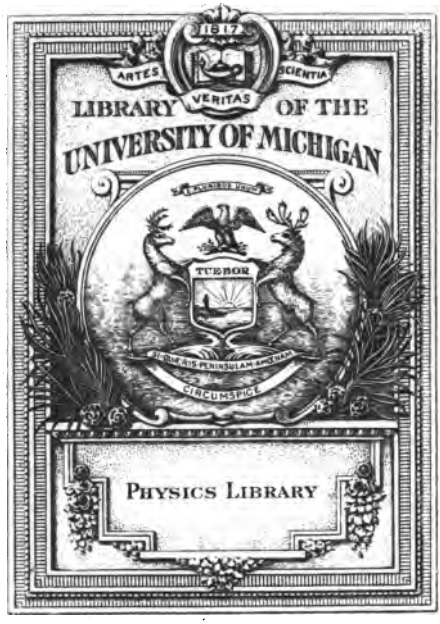
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THEORY OF HEAT.

BY

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PREFACE.

THE following Treatise is an attempt to present in a single view, the most important facts at present known on the subject of heat, and to deduce a theory which, if it does not comprehend all the phenomena, is at variance with none. The French writers on the subject have adopted the title "Theory of Heat" to express the mathematical solution of the problems of Radiation and Conduction, but with the exception of M. Ampère's suggestions, no attempt that I am aware of has been made to combine the varied and apparently opposed characters which present themselves to our view in regarding heat as the agent of expansion, whilst it exhibits all the properties of a state of vibration. The hypothesis which makes heat a *substance* is utterly incapable of accounting for phenomena analogous to those of light, at least in a manner at all consistent with the explanation (now generally deemed correct beyond dispute) of the parallel properties, whilst on the other hand the supposition of a vibratory motion appears to be totally inconsistent with the facts of latent heat and expansion. The former being a Statical hypothesis is

inadequate to take cognizance of the effects of position, otherwise than by reference to the direction in which the equilibrium is disturbed; the latter being a Dynamical hypothesis, fails altogether to represent actions, the operation of which, whilst it varies with a change of circumstances, is totally uninfluenced by time.

In addition to the circumstance that no theory of heat has yet been promulgated, sufficiently comprehensive to include the explanation of all the phenomena, even the simple and well known facts relative to pressure and density are felt to be utterly inexplicable on any of the ordinary suppositions, without the aid of a force, the effect of which is insensible at a finite interval. It is true Mr Dalton appears to have recourse to no such hypothesis, but I do not know that he conceives pressure to be due to the actual contact of particles, and without such a supposition I am at a loss to imagine the manner in which he accounts for the phenomena. Nothing however can be more lucid than his ideas of the constitution of bodies; and I beg to acknowledge that I owe the groundwork of my theoretical explanations of this branch of the subject entirely to him.

It may be expected that I should give a reason for omitting the explanations of the phenomena of expansion, &c., given by Laplace in the Mécanique

Céleste. My principal reasons are that the law of force which Laplace assumes, is one which I cannot *conceive*, having nothing analogous by which to represent it to the mind; and that if I were to allow the possibility of such a law of force, I should be disposed to question many parts of the process by which his conclusions are obtained; particularly the investigations in V. 5. pp. 105, 106, 107, &c. In several memoirs in the Transactions of the Cambridge Philosophical Society have appeared my reasons for supposing that molecular forces are forces varying according to the law of the inverse square of the distance, it is not therefore necessary to enter at all into that subject at present. The hypothesis which I have adopted is that of Dalton, viz. that particles of matter are surrounded by particles of caloric; the magnitude of the latter being very much smaller than that of the former. To this supposition I have attached that of a mutual repulsion between particles of the same medium, and a mutual attraction between those of an opposite one, which furnishes me with a very simple explanation of the facts relative to pressure and expansion, by rendering it necessary that the particles of caloric should be considered as actually in contact with, and pressing against those of matter. By this means the law of the inverse square of the distance is as probable as any other, whilst Dynamical considerations, at the same time that they lead

obviously to the law, afford a complete solution of all the difficulties attending on the phenomena of radiation and transmission.

It has been my endeavour, if possible, to introduce no *new* supposition, but rather to shew that those which I do not find ready to my hands may be arrived at as necessary consequences of the general law: by this means the whole is rendered the result of one grand principle; and if it should be found that some points of the theory are rather indefinite and capable of almost any interpretation which may be required, my defence is that we still labour under a degree of uncertainty as to the experimental facts, and consequently cannot limit our theory to a narrower bound than such as will include all the probable results.

To give an instance. We have no very definite idea of the meaning to be attached to the word temperature: that it is a measure of the state of heat within a body there can be no question, but in what way increments of temperature correspond to increments of heat we know almost nothing at all Whether as Dalton supposes, all bodies expand in geometrical progression, as the quantity of heat they contain is increased in arithmetical progression, or whether bodies of a consistence least likely to be affected with deviations due to changes in the distance of their particles, (such as the gases) expand equally

by equal increments of temperature, is a subject on which we can at present form no correct conclusions. The former hypothesis is an extremely probable one, and it consequently behoves us to leave sufficient latitude in our assumptions by which the latter is explained, to admit of a change to the former, should it be found requisite. But whilst it is necessary not to limit the detail of our hypothesis, it is highly incumbent on us not to admit of assumptions which are in the slightest degree inconsistent with each other. It has consequently been my especial aim to effect unity of hypothesis, preferring a colourable explanation of some difficulties to a disconnected solution of all, and leaving open a considerable number of phenomena, which a very slight addition to the first principles would have reduced to theory, whenever it appeared probable that subsequent consideration would render such addition needless.

It will be seen that I divide the subject into two parts, Statical and Dynamical. The latter is based on calculations suggested by the properties of light, and the two are connected in the following manner.

If heat is to be transmitted from one point to another across any substance, the state of the system is evidently such, that there is a number of particles on the outside of the surface, greater than would be the case were the system in equilibrium; the consequence is, that the repulsion from without communicates a state

of motion to the particles within, analogous to that with which they themselves are affected. But in addition to a motion of translation forwards, the particles may be endued with a transverse vibration, which vibration will neither affect nor be affected by the mean motion of transmission forwards, provided the medium is symmetrical and continuous. Now at the surface this continuity ceases, and also an equilibrium amongst the particles no longer exists; at this point, as it appears from calculation, the particles will receive a new motion of transmission, the magnitude of which varies as the sum of two expressions; the one being the excess of repulsion due to the aggregation of particles; the other, a quantity depending on the *vis viva* of the vibratory motion. The analytical investigation on which this depends is by no means complex, but to those who dislike the trouble of consulting it, I explain it in the following manner. Suppose the particles arranged in a square order, that is, that eight particles of the medium are in the corners of a parallelopiped, then if a set of them, all lying in one plane be simultaneously moved in that plane, the obvious result is, that the repulsions which they experience will be altogether changed; if the medium be symmetrical on both sides of this plane, this change of force will produce no effect, the additional repulsions being equally balanced; if this be not the case, it is obvious that the pressure

forwards will be changed in the same proportion as the forces are changed.

By the introduction of this mode of reasoning, which involves no new principle, but follows as a necessary consequence of the Statical hypothesis, we are enabled to account for a numerous class of phenomena, and to reconcile the apparently conflicting facts of expansion and polarization.

With respect to the remainder of the work, as I have given no authorities, it may be requisite that I should offer a few observations on the sources from which I have derived my materials.

The law of radiation is proved by Leslie in his *Inquiry into the nature of heat*, by experiment: in the same treatise will be found a variety of interesting experiments connected with the subject of polish, the nature of the radiating surface, &c. The theoretical explanation is that of Fourier, *Memoirs of the Institute*, 1821. Other facts of observation will be found in the *Annals of Philosophy*, V. 23, &c. and in the *Annales de Chimie*, and *Philosophical Transactions;* for a complete account of which I cannot do better than refer to Professor Powell's able Report to the British Association for 1832. With respect to the laws of cooling, they are the results of the experiments of MM. Dulong and Petit, and are to be found in the *Annales de Chimie*, V. 7.

In the Chapter on Conduction I have followed M. Fourier, in his *Théorie de la Chaleur*, and in the *Memoirs of the Institute*, 1819, 1820, in preference to M. Poisson's investigations for two reasons: in the first place, because I consider the principles on which M. Poisson deduces his equations devoid of simplicity; and secondly, because the complete solution of the equations is a subject of such extreme complexity as to be out of place in an elementary work like the present. The few examples which I have given of discontinuous functions and their verification, rest on my own responsibility, as does also the investigation of the motion of heat in the ring on the assumption of Dulong and Petit's law of cooling: the reasons for my rejection of M. Libri's solution will be found in the work. For further information on the subject of Conduction, I refer to Mr Whewell's Report before the British Association, 1835.

For information on the subject of Gas and Vapour the reader may consult the *Manchester Memoirs*, V. 5, and Dalton's *New System of Chemical Philosophy*; the *Annales de Chimie*, V. 5 and 10; the *Journal de Physique*, V. 89; and the *Annales de Chimie*, V. 35 and 41; Dr Wells's Essay on Dew, and various articles in the *Annals of Philosophy*.

The experiments on Polarization are exclusively those of Melloni and Forbes; the former are described in the *Annales de Chimie*, 1833, &c.; the

latter in the Transactions of the Royal Society of Edinburgh.

The subject of Terrestrial Heat is treated of in the Memoirs of the Institute; in the *Bulletin des Sciences*, 1820; and in the *Connoissance des Temps* for 1823. The experimental part will be found in Poisson's *Théorie de la Chaleur;* in the *Transactions of the Geological Society of Cornwall*, &c.; and in Daniell's Meteoreological Essays.

On the subject of Theory the writings of Dalton, Avogadro and Ampère are the most important, the last are to be found in the *Annales de Chimie*, V. 90. In Dr Prout's *Bridgewater Treatise* will also be found some curious speculations on the subject, and in a late number of the *Scientific Memoirs* is the translation of an Essay by M. Mossotti on the forces which regulate the internal constitution of bodies. In the introductory exposition, the author proposes to determine whether attractive and repulsive particles, such as those assumed by Franklin, may not by their mutual actions preserve a state analogous to that which we find in bodies. The investigation, however, valuable as it certainly is, proceeds on a totally different ground, depending on two assumptions, viz. that the *pressure* of the particles of ether varies as the square of the density, and that the difference of attraction on two sides of a material particle is in general very much greater than the difference between

the extra attraction of the caloric about the other particles, and the repulsion of those particles, an assumption which is altogether inconsistent with the supposition that the law of force is that of the inverse square of the distance. In fact, M. Mossotti tacitly supposes that the principal effect is due to actual *pressure* of the particles of ether on the material ones, a view of the subject which exactly coincides with my own, as I enunciated it in the *Transactions of the Cambridge Philosophical Society*, V. 6, p. 261.

The reader will find details of experiments on specific heat in Dr Black's Lectures, Irvine's Chemical Essays, the *Memoirs of the Academy of Stockholm*, the *Mécanique Céleste*, in the *Annales de Chimie*, V. 85, by Delaroche and Berard, and in Vols. 7 and 10 by Dulong and Petit, in Dalton's *New System of Chemical Philosophy*, and in Biot's *Traité de Physique*.

On Latent Heat, he may consult Dr Black's *Lectures*, and Biot's *Traité de Physique*, Vols. 1 and 4.

A discussion of the claims of the respective authors has been deemed superfluous; the reader will find the subject ably treated in Whewell's *History of the Inductive Sciences*, V. 2, to which work I beg to refer him for historical information on the subject.

QUEENS' COLLEGE,
December, 1837.

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THEORY OF HEAT.

CHAPTER I.

PROPERTIES OF HEAT.

1. When we use the term *heat* in common language, we express our sensations experienced in touching or coming near bodies differently circumstanced from ourselves; and a body which communicates a certain sensation to us is called a *hot* body. In a strict sense the word *heat* is used to denote the agent whereby the effect is transmitted; and the popular (probably the correct) idea attached to heat or *caloric* is that of a subtle fluid emanating from hot bodies, and entering between the particles of colder ones.

Some philosophers have conceived that the transmission of heat from point to point is not the transmission or the cumulation of a substance, but simply the transfer of a state of relative motion of the constituent particles. We will not attempt at present to discuss the merits of either hypothesis, but in order the more readily to understand terms which have been invented to express the effects of peculiar operations, we will adopt the former notion, and speak of heat as a subtle imponderable fluid pervading all material bodies, and manifesting its presence in them by the expansion which it causes in them, or in some other body in contact with them.

2. When a warm body is immersed in a cold liquid, it is found that the former loses and the latter acquires heat, and the performance of converse operations shews that the one loses exactly as much as the other acquires:

which affords strong reasons for adopting the view we have taken of the nature of heat, in preference to purely undulatory considerations.

3. The sense of feeling is a very imperfect judge in the case of heat and cold. Some idea, indeed, may be gained by it as to the quantity of heat which a substance communicates to the body in a given time, but none whatever as to the state of the body itself. Thus, a mass of iron and of wood touched by the hand, present very different sensations, the former appearing more cold than the latter, whilst it may happen that the converse is the actual state of the bodies themselves, independently of their readiness in parting with their heat.

4. We are then compelled to seek for some more accurate measure of the presence of heat or free caloric. No simple process could be expected to indicate the quantity of caloric absolutely contained in a substance, we shall be satisfied with a mode of determining its excess or defect in different bodies.

It is found that all bodies expand by heat and contract with cold, provided they do not experience a change of *consistence.** When such a change takes place the converse often happens. Thus, water contracts till within a certain distance (about 4.1^0 cent.) of the freezing point, and then commences to expand. The same is true of many other, but not of all fluids. Mercury for instance contracts in freezing.

To express the state of heat in a body we adopt the word *temperature.*

5. In order the better to understand the signification of this term, it is necessary to explain the thermometer.

This instrument is founded on the property that substances expand with heat and contract with cold, i. e. an

* I have adopted this term from Mr Whewell, History of the Inductive Sciences, vol. II. p. 499.

abstraction of heat. The common mercurial thermometer consists of a bulb with a very slender tube rising out of it. The bulb and part of the tube are filled with mercury, the remaining part being a *vacuum*.

The temperature of melted ice is constantly the same, if the water be perfectly pure; if the instrument therefore be plunged into melting ice, and the altitude of the mercury in the tube marked, this point constitutes a proper point for the commencement of graduation, and is in the centigrade thermometer, the *zero* point of the graduation. Fahrenheit's scale, however, which is the one chiefly used in this country, has 32^0 at this point, and terminates in 212^0, where the centigrade marks 100. This latter point is the point of ebullition when the barometric column is 29.9.

Should the barometer not stand at this precise height when the instrument is made, a correction is required, owing to the fact, that water boils at a less temperature with a small pressure than with a great.

6. It is worthy of remark, that however accurately the instrument be made and graduated, it will after a short interval be found incorrect, for if plunged into melting ice the surface is *above* zero. At the end of two or three years this elevation sometimes attains about 2^0; and after this interval it does not alter. This circumstance appears to arise from a mechanical constraint with which the particles are affected by the sudden transition from heat to cold, &c. in making the instrument; which constraint is rectified by regular changes of temperature continued for a considerable space of time.

The space between 0^0 and 100^0 is divided into equal volumes, and the number of degrees corresponding to the surface of the mercury is the measure of the temperature.

7. Had the instrument been made with alcohol or water, or any other liquid, the degrees marked under any

circumstances by these thermometers would also be styled temperature, but two thermometers of different liquids would by no means coincide in corresponding circumstances; thus, when the mercurial thermometer stands at 75^0, the water thermometer stands at 57^0.

It is evident from this circumstance that the temperature as marked by degrees on the thermometer is not a measure of the quantity of heat introduced. Had we a substance which expanded proportionably to the quantity of heat, we might use this substance as an accurate measure of temperature. Atmospheric air and gases possess this property, and hence the air thermometer possesses advantages over every other. The general rule is that substances expand more at a high temperature for a given increase of heat than at a low. It happens fortunately that the expansions of glass between the limits of the melting and boiling points vary just in the same ratio as those of mercury, and thus the want of uniformity in the expansion of the mercury is compensated for by the want of uniformity in the expansion of the instrument.

8. Air expands much more by heat than fluids, so that an air thermometer possesses the advantage of delicacy in addition to that of accuracy. But as the expansion between any considerable limits of temperature would be very large it could not be easily measured. To remedy this defect an instrument called *The Differential Thermometer* has been invented. It consists of two bulbs each filled with air, connected by a tube bent twice at right angles, so that each bulb is at the top of its connecting tube, and this tube contains a quantity of coloured sulphuric acid. So long as both bulbs are affected with the same temperature the acid is at rest, but when one experiences more heat than the other the acid moves towards the latter. By this means very slight differences of temperature can be detected; and this instrument has the advantage of being unaffected by atmospheric pressure.

9. The contrivances which we have hitherto described serve to measure with more or less accuracy any quantity of heat which is not excessively small. Another instrument invented by M. Nobili, detects the most minute accessions of heat, and is one of the most important inventions of the present age.

If a bar of bismuth be soldered to one of antimony, and a small quantity of heat be applied at the point of junction, a galvanic current is immediately produced between the metals, the intensity of which can be determined by means of a galvanometer. M. Nobili took advantage of this property to measure extremely small quantities of heat in the instrument called the *Thermo-Multiplier.*

This instrument consists of a pile of bars of bismuth and antimony soldered together, so that one extremity of a bismuth bar is soldered to the antimony bar above it, and the other to that below it, and so on, forming a complete galvanic battery, which is acted on by heat at the soldering points. The extremities of the pile are connected by a long chain of brass wire covered with silk, which is twisted many times round a box which contains a magnetic needle, in such a manner that the wire is parallel to the needle in its state of equilibrium.

As soon as heat applied to the pile causes a stream of galvanism to pass through the wire, it is evident that the needle will be affected, and we can calculate the intensity of the heat by the deviation of the needle.

10. When very high degrees of heat are to be observed we must make use of some other substance, since fluid vaporizes at high temperatures and are otherwise subject to error. The instruments used for this purpose are called *Pyrometers.*

Wedgwood's pyrometer was founded on the principle that clay (a compound of earth and water) gradually loses its water when exposed to a high increasing temperature,

and consequently contracts. As however the use of this mode requires a separate piece of clay for every observation it cannot be relied on.

Daniell's pyrometer consists of a black lead bar in which a hole being drilled, a small bar of platinum is fixed at one extremity, whilst the other, which is within the hollow tube of black lead, rests on a small cylinder of earthenware which, by its expanding more than the black lead case, it pushes out. From the position to which it has been pushed, we ascertain how much the platinum has expanded more than the black lead and earthenware, and then could, were the expansions uniform, calculate the heat.

11. We have now a method by which the meaning of the word temperature can be expressed. Thus two substances are said to have the same temperature when a thermometer is equally raised by remaining in contact with either of them, and the degree of elevation due to any particular body is called its temperature.

12. When in the following work we speak of the temperature in degrees, we refer to the centigrade thermometer. The results can readily be transferred to another scale—thus in Fahrenheit the freezing point is 32^0, and the boiling 212^0—it is clear therefore that a degree of the centigrade is equal to $\frac{180}{100}$ or $\frac{9}{5}$ of a degree of Fahrenheit: hence if the centigrade thermometer indicate a degrees, Fahrenheit's will indicate $32 + \frac{9}{5}a$ degrees. In the same manner a number of degrees on Fahrenheit's scale can be transferred to the centigrade by diminishing them by 32^0 and taking $\frac{5}{9}$ of the remainder.

13. *Specific Heat.* The increase of temperature of a body is no criterion of the quantity of heat communicated to it, as it is found that a given volume of one substance

is very differently affected by a certain quantity of heat from a given volume of another. If equal quantities of water at 20° and 30° be mixed together, the temperature of the mixture is 25° nearly, so that speaking popularly, the quantity of heat which one has lost in falling from 30 to 25 just suffices to raise the other from 20 to 25, that is through the same number of degrees.

But if a pint of mercury at 100° be mixed with a pint of water at 40°, the mixture will have a temperature of 60°, so that the 40° of heat lost by the mercury only suffices to raise the water 20°.

If equal *weights* of mercury and water be mixed the result is still more striking. Thus a pound of mercury at 160° mixed with a pound of water at 40 will give the mixture a temperature of about 44, shewing that the same quantity of heat which raises water 4°, raises an equal mass of mercury 116, or about thirty times as much.

The specific heat of body is estimated by the quantity of heat requisite to raise its temperature from a given height α to another given height β, and is generally estimated by supposing equal *weights* of the substances raised in the same manner. Thus if the specific heat of water be taken at 1, that of mercury is found to be .033, or about $\frac{1}{30}$th of the former.

14. There are various other circumstances connected with specific heat, among which are the following:

1. Change of *consistence* is accompanied with change of specific heat.

2. The specific heat of gases is not easily ascertainable, nor is it indeed measured exactly in the same way as that of solids. We cannot compare the quantities of heat requisite to raise equal weights of the gases by the same number of degrees; but must have recourse to volumes, and thus we shall have two cases to consider: (1) the specific heat

when the pressure is constant as when the gas is submitted to the atmospheric pressure by the action of mercury, the space in which it is contained expanding as the volume extends: (2) the specific heat when the volume remains unchanged, as when gas is confined in an inextensible vessel.

In the former case it is believed that the specific heat is the same for all temperatures.

In the latter case experimenters have come to opposite conclusions: this only is known, that if two equal weights of the same gas be subject to such pressure, that one occupies double the space of the other, the specific heat of the former is greater than, but not twice as great as that of the latter.

3. The specific heat of solids and liquids is greater at a high than at a low temperature; which is probably owing to their dilatation.

4. Dulong and Petit have observed a curious analogy between the specific heat and the atomic weight of bodies, viz. that the product of the specific heat by the atomic weight is a constant.*

15. The usual method of determining the specific heat of a substance is by mixing it with some other substance whose specific heat is known.

Thus, if s be the specific heat of the substance required,

s'.....of the standard substance,

* The atomic weight of a substance is the weight of an atom on the supposition that a simple substance may be represented as a congregation of atoms all of the same dimensions. Compound substances in a state of chemical union are so formed as that a given number of atoms of each substance is united in a simple ratio: thus, water consists of oxygen and hydrogen mixed in such proportions that the weight of the oxygen is eight times that of the hydrogen: if we suppose water composed of one atom of the one, to one atom of the other, this will give the weight of an atom of oxygen eight times that of an atom of hydrogen. It is, however, perfectly conceivable that water is composed of two atoms of oxygen to one of hydrogen, which would give the atomic weight of oxygen four times that of hydrogen instead of eight.

w, w' their weights; t, t' their temperatures,

θ the temperature of the mixture,

$t - \theta$ is the temperature lost by the substance;

$\therefore$ $s(t - \theta)$ is the heat lost by an unit (of weight),

and $\therefore$ $ws(t - \theta)$ is the actual heat lost.

Similarly, the standard substance gains $w's'(\theta - t')$. Now what one substance loses the other gains;

$$\therefore ws(t - \theta) = w's'(\theta - t');$$

$$\text{hence } s = \frac{w}{w'} \cdot \frac{\theta - t'}{t - \theta} \cdot s'$$

is the specific heat required.

16. This method is inapplicable to gaseous bodies. Crawford's method of determination of their specific heat consisted in plunging two similar vessels, one filled with the gas, the other exhausted, into hot and then into cold water. The quantity of heat given to the cold water by that vessel which contained gas being greater than that from the other, the difference of course gave the specific heat of the gas corresponding to a constant volume. This is not much to be relied on owing to the small *difference* compared with the total quantity of heat. Various other methods have been adopted,—that of the calorimeter consists in introducing heated gas into a tube surrounded by ice, whence by knowing the temperature at which it entered and that which it has when it has passed through the instrument, and weighing the quantity of ice melted, the specific heat is known. In this case the heat is not that which would raise the temperature from the latter observed height to the former whilst the volume remains constant, since the volume will change during the experiment.

17. *Latent Heat.* When a substance passes from one state to another, as when a solid is melted or when a fluid evaporates, a considerable quantity of heat is either evolved or absorbed.

Thus, in order to melt ice at the temperature 0^0, so that the water itself shall have the same temperature, as much heat is expended as would raise the water from 0^0 to 78^0; and again in passing into vapour as much heat is expended as would raise water to 528^0.

This loss of heat is called its *heat of fluidity*. From the circumstance that heat exists in the fluid which is insensible to the thermometer, the term *latent heat* has been applied to the unknown quantity of heat which bodies thus contain. Dr Black invented the term to express the loss of heat mentioned above, but it is objectionable owing to the circumstance, that this loss is only the *difference* between the latent heat of water and that of ice.

18. We can easily determine the heat of fluidity for a substance which freezes above 0 by means of the calorimeter. Let t be the temperature at which the body freezes, $t + t_1$ any other temperature, to which let the body be raised and then placed in the calorimeter. Let s, s_1 be its specific heat in its fluid and solid state; M its mass.

Now when placed in the calorimeter enclosed by ice it will cool down to 0^0.

The quantity of heat which it loses before it reaches the temperature $t = Mst_1$, and the quantity after $t = Ms_1t$: hence the whole quantity $= M(st_1 + s_1t)$. If the quantity gained by the ice be calculated from the quantity of ice melted it is found $= c$ a greater quantity than the above; the difference $c - M(st_1 + s_1t)$ is the quantity lost suddenly in the change of consistence which is what we require.

19. Not only is heat lost to the thermometer when a solid melts, but a very considerable quantity also becomes latent when a fluid evaporates. In order to estimate this quantity Count Rumford employed a means analogous to that which we noticed as used to determine the specific heat of a gas.

In a calorimeter filled with water at a temperature less by a than that of the external atmosphere t, is made to circulate the vapour of a liquid having the original temperature T. By this means it loses its latent heat, liquefies, and is withdrawn from the instrument at a temperature whose mean value is t; the experiment being stopped when the temperature of the calorimeter becomes $(t + a)$.

Let s be the specific heat of the liquid; M the mass of water in the calorimeter, added to a quantity which is equivalent in absorbing of heat to the vessel itself; m the mass of vapour, L the latent heat in an unit of vapour;

t' the temperature at which the substance vaporizes;

then quantity of heat lost by vapour $= ms(T - t')$;

also that lost afterwards by fluid $= A(t' - t)$;

that gained by the water is $2Ma$; where A is known from the specific heat of the fluid;

$$\therefore 2Ma - ms(T - t') - A(t' - t),$$

is the excess of the quantity gained by the water above that lost before liquefaction and after, it is therefore $= mL$, the quantity required.

The latent heat of vapour is such, that if it be added to the thermometric heat of the water, it produces the same constant result at all pressures and temperatures; this result is 643^0.

Some philosophers assert that this is only true for a given temperature, whilst M. Dulong is satisfied that it is not accurately true at all.

20. *Radiation.* The way in which bodies part with their heat from their surfaces is termed their radiation, from the similarity which exists between the rays of heat and rays drawn from a center. These rays pass freely through the air, and on meeting with the surface of a body, they gradually raise its temperature, and the degree

of rapidity of this increase of temperature depends on its absorbing power. If a body suffer heat to pass very freely through it, as Rochelle salt does, its own temperature will of course be affected slowly: the same will happen if the surface of the body reflect heat very copiously. The law of reflexion for heat is the same as that for light, viz. that the angles of reflexion and incidence are equal. It is reasonable to conclude from the analogy which heat bears to light, that the reflecting power should depend both on the nature of the substance itself, and on that of the surface on which the heat is incident. The more accurate the polish, the more copious the reflexion.

It is found by experiment, that the same circumstances which conduce to perfect the reflexion, tend to diminish the radiation: thus, metal surfaces polished, radiate slowly but reflect copiously: whilst the same metal with a blackened surface radiates copiously, but reflects little. The law of emanation in heat as in light is, that it varies as the sine of the angle of emanation, that is, that the intensity of emanation from any point in a direction making a given angle with the normal, to the surface at that point, varies as the sine of this angle. If two mirrors be placed in such a manner, that the focus of one shall be conjugate after two reflexions, to that of the other, that is, that rays radiating from one focus shall by means of two reflexions converge to the other, it is found that if a hot body be placed in one, and the bulb of a thermometer in the other, the thermometer instantly rises: on the other hand, if a ball of ice be placed in the first focus the thermometer falls.

In the former case it is evident that the whole number of rays which fall on the aperture of the mirror, are again collected into one point on the bulb of the thermometer; in the latter, it appears that heat is proceeding towards the ice in every direction; the quantity therefore which leaves the mirror next it, is supplied from all directions; and hence the second mirror sends its heat to the cold

body. An equal quantity to that which it thus loses must be supplied to it from the surrounding medium—but there is only one point in the surrounding medium from which that supply will come in such a direction as to be reflected from it in such a manner as to supply the heat abstracted —that point is in the bulb of the thermometer: it is clear therefore, that a considerable supply of heat must proceed from the thermometer to the mirrors.

21. The subject of radiant heat is one of the most extensive, involving the considerations of reflexion, absorbing power and the like. As a subject for mathematical investigation it presents little to our notice, from the circumstance that we are unacquainted with the physical constitution of bodies which leads to their peculiar effect on the heat which falls on them, or passes through them.

We shall mention some of the results of experiment on this branch of the subject in the chapter on radiant heat.

22. *Communication of heat.* When we say that a body placed in contact with another acquires the temperature of the other, we speak only under a popular point of view; the fact is, the body of greater temperature will communicate a portion of its heat more or less rapidly to the other, and in so doing will have its own temperature diminished. We of course do not take into the account its loss of heat by communication with the air. The communication of heat by contact is the more rapid as the contact is more perfect. It is true, heat is transferred slowly to a gas, but this may arise from another cause. The transfer from a solid to a liquid is more rapid than from a solid to a solid, and from a liquid to a liquid it is still more rapid.

23. These circumstances render the senses very inadequate criterions of the temperature of a substance. The hand can touch glass with impunity at a temperature to which if iron were heated it would be impossible to bear it; in

like manner cold iron gives a much more vivid sensation than glass of the same temperature. Metals when melted and at a white heat, will not burn the skin as when colder.

24. *Conduction.* It has been stated that when a body is hotter than the surrounding bodies, it loses heat gradually from its surface by radiation. If, however, instead of making the whole body hot, we heat part of it, we find that heat is communicated to the other parts, this communication of heat is called conduction.

Different bodies have different conducting powers depending to a certain degree on the compactness of their texture. Thus, the conducting power of gold is nearly three times that of iron, whilst that of iron is double that of lead. Count Rumford found that the same substance might be made to conduct very differently when differently employed. The warmest clothes, such as wool, fur, &c. are the most imperfect conductors.

25. *Convection.* A mode of loss of heat analogous to conduction is that to which Dr Prout has applied the term *convection.* When a hot body is in contact with the air, the part next the body becoming more elastic, flies off and is supplied by colder portions; thus, the heat of the body is converged away more rapidly than it would be if the air were not in motion. It is obvious that this circumstance will materially affect all experiments on the motion of heat, in which it is hardly possible to estimate the effects due to this cause.

26. *Sources of Heat.* Under this heat we range all those circumstances which though they cannot be properly said to give rise to heat, yet manifest its appearance to the senses. We do not confine ourselves to the heat of the sun, of electrical action and of chemical decomposition, but consider mechanical causes which develope heat as equally sources with the above.

The first point to be observed is, that the earth derives its heat from the sun, on which account we are to look on that body as the grand source of heat. It is not however necessary to suppose that the sun is continually emitting rays of heat which are accumulating on the earth and spreading in space. The most probable hypothesis with respect to the *transfer* of heat is, that none takes place except from a body which contains more sensible caloric than another, in a direction towards that other. Thus it is possible that there is a reciprocating motion of caloric to and from the strata of the atmosphere which, if they influence the sun at all may at some periods of the year (near the equinoxes) communicate heat to him. On this supposition the sun would be looked upon as a mere exciting cause.

27. When the heat of the solar spectrum is examined it is found that the position of greatest heat is different according to the nature of the substance employed to analyze the light. In water, it lies in the yellow space; in sulphuric acid, it is in the orange; in crown and common glass, it is in the middle of the red: and in flint glass, it is beyond the red. In all cases there is a perceptible heat at some distance beyond the visible boundary of light.

We are not warranted in concluding from the above that the rays of heat are separate from the rays of light. It is however probable that the whole effect is the sum of two separate effects; the one corresponding to heat viewed independently of light, the other arising from the light itself; and the way in which the former is affected by the substance will determine the position of greatest heat. Sir D. Brewster assumes that the heating powers of the sun's rays, taken as they come from the sun, varies as their respective illuminating powers. From the circumstance that the hottest part of the water spectrum is the yellow, which is also the brightest, it follows that water absorbs the solar *heat* in the same proportion as it absorbs the solar *light*. The other causes of the production of heat present nothing particularly worth discussing.

CHAPTER II.

RADIATION.

28. We have already slightly touched on this subject, it remains that we point out some of the most remarkable variations in radiation as connected with observation. To a certain extent it will be necessary to unite the radiating power of a substance with its conducting power, from the fact, that in many instances we are unable to say precisely to what extent the effects which we observe are due to the nature of the surface, or to the state of sudden change from the surface to surrounding bodies.

29. That heat radiates in a vacuum is proved by the following experiment. A hollow globe of glass has a thermometer fixed to it, so that the end of the tube alone is in contact with the globe. The air is withdrawn by affixing to the globe a long tube opening into it, and plunging the apparatus in mercury, so as to fill it. On withdrawing the globe and part of the tube from the mercury, the globe together with that part of the tube to which the mercury does not rise by the atmospheric pressure, is a vacuum. Cut off the communication between the globe and tube, by passing the flame of a lamp across it, by means of a blow-pipe; let this apparatus be plunged into boiling water, and the mercury in the thermometer rises rapidly. This rise is not due to the conduction of heat along the thermometer tube, since glass conducts heat very slowly —it is consequently the effect of heat radiated across the vacuum, between the globe and bulb of the thermometer.

30. By experiment it appears that the quantity of heat received in an unit of time by the same body at different distances varies inversely as the square of the distance.

This result is consistent with the following considerations.

If heat be supposed to consist of lines of particles equally moving in all directions from a hot point, it is obvious that the same number of particles will in the same time pass through any two spherical surfaces described about that point. Now the area of the spherical surface varies as the square of its radius,

Let therefore r, R, be the radii of the surfaces,

a, A, an area in each of them respectively,

$$\text{then } \frac{\text{quantity of heat received by } a}{\text{..... whole surface}} = \frac{a}{\text{whole surface}} = \frac{a}{cr^2},$$

$$\frac{\text{quantity.....} A}{\text{..... whole surface}} = \frac{A}{cR^2};$$

therefore if $a = A$,

$$\frac{\text{quantity of heat on given surface at distance } r}{\text{..... } R} = \frac{\frac{1}{r^2}}{\frac{1}{R^2}}$$

which is the experimental result.

31. Another conclusion from experiment is, that the quantity of heat received on a given plane at a given distance, but inclined to the direction in which the heat proceeds, varies as the sine of the angle of inclination of that direction to the plane.

To prove this conceive the plane very small, or the radiating point very distant. Then the same number of rays which are received by the plane in its oblique position would be received by another plane perpendicular to the di-

rection of radiation, whose area is the projection of the given plane on the latter.

Hence, intensity of heat on a given plane : that on a perpendicular plane :: projection of given plane : given plane :: sin $\angle$ of inclination of plane to direction of radiation : 1; which is the proof required.

32. Thus far we have given some of the general properties of heat as influencing the body on whose surface it falls; it is necessary next to consider the effect of varying the position of the surface from which it emanates. The law deduced from experiment is that the intensity of emanation in any direction is proportional to the sine of the angle which the direction of emanation makes with the surface.

This law is thus established on theoretical grounds.

33. PROB. To find the law of emanation of heat from the surface of a hot body.

Fourier's hypothesis is that the rays radiate not from the surface alone, but from a small distance beneath the surface; we shall see in the sequel that this hypothesis is only introduced for the purpose of computation, as the depth to which the radiation is sensible, as well as the function of the depth according to which the intensity of emanation varies, disappear in the final result, and are therefore perfectly arbitrary.

Let a be the perpendicular distance of any point within the substance from the surface; ϕa the function which expresses the quantity of heat emitted by this point through the surface, a the distance at which this radiation becomes insensible; then $\int_0^a \phi a \, da$ is the quantity emanated from a perpendicular line of particles: call this Q.

But in a direction making the angle ϕ with the surface, the distance of the point from the surface or the space through which the heat traverses is $\frac{a}{\sin\phi}$; also the area of the exterior surface through which this emanates must remain the same as before; hence the quantity radiated by the portion between two planes parallel to those taken hypothetically in the last case to include an element is $\phi\left(\frac{a}{\sin\phi}\right)da$, and consequently, the whole quantity is,

$$\int_0^a \phi\left(\frac{a}{\sin\phi}\right)da.$$

Let $\frac{a}{\sin\phi}=\beta$;

then this quantity is $\sin\phi\int_0^{\frac{a}{\sin\phi}}\phi\beta\,d\beta$.

Now $\frac{a}{\sin\phi}>a$, and $\because\ \phi(a)$, and the same function of any quantity greater than a is $=0$; it is evident that

$$\int_0^a \phi\beta\,d\beta=\int_0^{\frac{a}{\sin\phi}}\phi\beta\,d\beta,$$

$$\text{or } \int_0^{\frac{a}{\sin\phi}}\phi\beta\,d\beta=\int_0^a \phi a\,da=Q;$$

hence the quantity radiated in a direction making the angle ϕ with the surface $=Q\sin\phi$, Q being that radiated in the normal direction.

34. The above result is the law of emanation as deduced from the hypothesis of intra-molecular radiation. The same result will also appear from the following problems.

PROB. 1. To find the permanent state of heat in a particle placed in the center of the axis of a cylinder of indefinite length.

Let h represent the exterior conductivity of the cylinder and particle, the temperature of the cylinder being = 1.

A the particle; AM the perpendicular to the axis of the cylinder $= a$, $MP = x$, $AMP = \phi$, $F\phi$ the law of emanation, then the quantity radiated from an element whose area is $2\pi a dx$ to A is

$$\frac{2\pi a dx h}{a^2 + x^2} . F\phi.$$

Let therefore Q be the whole quantity received by the particle, supposing it to reflect none,

$$Q = 4\pi a h \int_0^x \frac{dx}{a^2 + x^2} F\phi;$$

$$\text{but} \quad x = a \cot \phi,$$

$$\therefore \quad dx = -\frac{a}{\sin^2 \phi} d\phi.$$

$$\text{Hence } Q = -4\pi h \int_{\frac{\pi}{2}}^{0} \frac{a^2}{a^2 + x^2} \frac{d\phi}{\sin^2 \phi} F\phi$$

$$= -4\pi h \int_{\frac{\pi}{2}}^{0} F\phi \, d\phi;$$

but the quantity emanated from the particle is $4\pi ch$, if c be its temperature.

Hence when the temperature is permanent and the particle receives no more than it emits,

$$4\pi ch = -4\pi h \int_{\frac{\pi}{2}}^{0} F\phi \, d\phi,$$

$$c = \int_0^{\frac{\pi}{2}} F\phi \, d\phi.$$

1. Let $F\phi = 1$;

then $c = \frac{\pi}{2}$.

2. Let $F\phi = \sin\phi$;

then $c = 1$.

Hence the hypothesis of equal radiation in all directions gives as a result, that the particle would acquire a greater temperature than that of its enveloping cylinder: whilst the law of sines gives the temperatures equal, a result more consistent than the former.

PROB. 2. A spherical body is placed in the center of a spherical heated envelope, to find the permanent state of heat in it.

Let the radius of the envelope be 1, that of the enveloped sphere r, and suppose r very small compared with 1. At T take any area of the small sphere $= \omega$. Let the line drawn from it to a point Q in the envelope make the angle ϕ with the tangent plane at T, which angle is very nearly the same as that made by the radius drawn to Q with the same tangent plane; the area of an elementary annulus of the envelope is,

$$= 2\pi \cos\phi d\phi;$$

hence the quantity of heat sent by it to the element ω is,

$$\omega h 2\pi \cos\phi \sin\phi d\phi;$$

for from any point, the quantity received by an oblique plane is to that by a direct one, as a direct plane is to an oblique one.

Hence the whole quantity received by ω

$$= \omega h 2\pi \int_0^{\frac{\pi}{2}} \sin\phi \cos\phi d\phi;$$

and the whole quantity received by the small sphere whose area is $4\pi r^2$ is

$$4\pi r^2 \omega h 2\pi \int_0^{\frac{\pi}{2}} \sin\phi \cos\phi = Q.$$

Also the quantity emitted by the particle ω on the elementary annulus at Q

$$= c\omega h\, 2\pi \cos\phi d\phi F(\phi);$$

therefore the whole quantity sent away by ω is

$$2\pi c\omega h \int_0^{\frac{\pi}{2}} F(\phi) \cos\phi d\phi;$$

and hence as in the last Problem,

$$Q = 4\pi r^2 \omega h 2\pi c \int_0^{\frac{\pi}{2}} F\phi \cos\phi;$$

$$\therefore \int_0^{\frac{\pi}{2}} \sin\phi \cos\phi = c \int_0^{\frac{\pi}{2}} F\phi \cos\phi.$$

Now it is requisite that $F\phi$ should equal 1 when $\phi = \frac{\pi}{2}$, let therefore

1. $F\phi = 1$;

$$\therefore \tfrac{1}{2} = c.$$

2. $F\phi = \sin\phi$;

$$\therefore 1 = c.$$

Hence the supposition of equal radiation in all directions gives the permanent state of temperature of the enveloped sphere only $\frac{1}{2}$ that of the enveloping one, whilst the law of sines requires that they should be equal.

We may therefore conclude that the law of sines is the probable law of nature: and numerous experiments of Leslie and others confirm it. Indeed the law was *enunciated* by Leslie from experiment.

35. We have it is true said nothing of the heat reflected from the surface of the particle, but as the emerging heat is similarly reflected, the only difference will be that the interchange of heat goes on more slowly; for if R be a quantity of heat coming in any direction on the surface, a parts of the whole will be reflected, and hence $R(1-a)$ is all that enters.

Similarly if a quantity r were emerging, $r(1-a)$ will be all that would escape: thus, $\Sigma R(1-a)$ enters, and $\Sigma r(1-a)$ emerges, so that if $R=r$ the results will be uninfluenced by the value of a.

35. Newton assumed as the law of cooling, that the loss of heat in a body is proportional to the excess of its temperature above that of the surrounding medium; when the difference of temperature does not exceed 10^0 or 12^0, this is sufficiently accurate, but by no means so when the difference of the temperature is great.

36. *Velocity of cooling.* Let q be the excess of heat in the body, θ the excess of temperature: then θ is proportional to q.

Now $-dq$ is the quantity of heat lost in the time dt; hence the quantity of heat which would be lost in an unit of time, supposing the rate of loss constant during the whole unit, is $-\frac{dq}{dt}$, but q is proportional to θ, hence the quantity lost is proportional to $-\frac{d\theta}{dt}$ which is called the velocity of cooling, and is in fact the change of temperature which the body would experience.

There is a difference of circumstances when a body cools in vacuo from those under which it cools in air: in the latter case, in addition to the quantity of heat sent off by direct radiation, a certain quantity is carried off by the circumjacent air. The law of cooling to which MM. Dulong and Petit have arrived is different from that of Newton, and from its simplicity demands attention as most probably the law of nature. The mode of calculation and observation adopted by these philosophers is the following.

Having observed the difference of temperature of the cooling body and the surrounding medium at equal intervals,

they perceived that the law of Newton which gives $t = A\left(\frac{1}{m}\right)^x$, where t is the difference of temperature, x the time, and m a number varying for different bodies is not exact. They then assumed the formula $t = A\left(\frac{1}{m}\right)^{\alpha x + \beta x^2}$ to represent the differ- of temperature, and determined α, β, and m.

The velocity of cooling then becomes known from the equation,

$$V = -\frac{dt}{dx} = t(\alpha + 2\beta x)\log m.$$

It is not assumed as probable that this is the correct law, but it is found to apply much better than the law that the velocity of cooling varies as the excess of temperature. Indeed this formula serves very well to interpolate values of the velocity not very distant from those known, and is consequently a very convenient one to adopt in calculations of the results of experiment.

37. The next step in the investigation is to examine the effect due to a change of circumstances.

1. When the bulbs of the thermometers differ the law of cooling is not affected.

Three thermometers of different sized bulbs were used, and the expressions for their respective velocities of cooling observed. It was found that the ratio of the velocity of cooling of thermometer A to that of B at a temperature (α) was the same as at any other temperature (β). The velocities themselves were very different for the different thermometers, being nearly in the inverse ratio of the diameters of the bulbs.

2. The nature of the fluid does not affect the law.

A matrass containing the fluid to be examined, had a thermometer placed in it by which the temperature was observed. It was found that the position of the thermometer in the matrass did not affect the height of the mercury in it. In the same manner as in the former case, the ratio of the velocities for different substances corresponding to the same excess of temperature in each, was found to be independent of that excess itself.

3. The nature of the vessel does affect the law.

The velocities of cooling of a glass and a tin sphere being observed, it was found that the ratio taken as above of the velocity for the glass sphere to that for the tin, increased as the excess of temperature diminished. Mr Leslie had come to a similar conclusion.

4. The form of the vessel does not affect the law.

Observations were made on the cooling of three tin vessels of the same volume, the first spherical, the second and third both cylindrical of different diameters. The variations in the ratios were not greater than would arise from errors of observation, and irregularities in the construction of the vessels.

38. Having established these general principles as a foundation, MM. Dulong and Petit proceeded to deduce the general law of cooling in the following manner.

They constructed two mercurial thermometers, one having a bulb of six centimeters diameter, the other of two. The former was used for high temperatures, the latter for low in order to shorten the duration of the observations, as the large thermometer would cool very slowly when its excess of temperature was small. The velocity of cooling of the latter is transformed to that of the former by means of a constant factor expressing the ratio between them, which is previously observed, and is found as we noticed above

to be very nearly as the ratio of the diameter of the small thermometer to that of the larger.

The enclosure in which the cooling was observed was a large thin copper globe, the neck of which AB being made plane was kept horizontal by means of a level.

The globe was plunged into a large wooden trough of water, retained by means of ice or vapour at a constant temperature, and the cooling of the thermometer within the globe was observed by means of its tube, which stood without the globe, and was surrounded by a glass cylinder, the space between this cylinder and the tube communicating with the globe by means of apertures in its base to which the tube was fixed, and which served as the stopper to the globe.

It is obvious that by means of a communication between this cylinder and an air pump, we can exhaust the globe, and if requisite, fill it with gas. The thermometer was heated to near the boiling point of mercury about 330^0, and the observations commenced at 300^0 or 250^0 according to circumstances, a certain portion of the temperature being lost in transferring the thermometer to the instrument, exhausting the globe, &c.

39. *On cooling in vacuo.* It is obvious that a perfect vacuum can never be attained by the air pump. In order, therefore, to obviate the errors arising from convection and other causes, observations were made in air of different densities, and a correction calculated for the effect of the air.

The following is a register of the observations for different temperatures of the water surrounding the globe: the first column marked (E) is the excess of temperature of the thermometer: V_0 is the velocity of cooling of the thermometer when the water is at 0^0.

V_{20} do. do. at 20, &c.

$E.$	$V_{0°}$	$V_{20°}$	$V_{40°}$	$V_{60°}$	$V_{80°}$
240°	10.69	12.40	14.35		
220	8.81	10.41	11.98		
200	7.40	8.58	10.01	11.64	13.45
180	6.10	7.04	8.20	9.55	11.05
160	4.89	5.67	6.61	7.68	8.95
140	3.88	4.57	5.32	6.14	7.19
120	3.02	3.56	4.15	4.84	5.64
100	2.30	2.74	3.16	3.68	4.29
80	1.74	1.99	2.30	2.73	3.18
60		1.40	1.62	1.88	2.17

40. These experiments establish the following law: The velocity of cooling of a thermometer in vacuo, for a constant excess of temperature, increases in a geometric ratio, whilst the temperature of the surrounding medium is increased in an arithmetic ratio.

In order from this experimental result to establish a formula for the velocity of cooling, we assume that the quantity of heat which leaves a body, is partly counterbalanced by the quantity which it receives from the surrounding medium. In fact, if $t+\theta$ be the temperature of the body, θ that of the surrounding medium, $F(t+\theta)$ the velocity of cooling when the surrounding medium is at zero; $F(t+\theta)-F(\theta)$ is the velocity of cooling under the conditions we have supposed.

By what has been shewn above, it is clear that $F(t+\theta)-F\theta$ must vary as a^{θ}, where a is a constant

depending on the nature of the surface, &c. of the thermometer.

But the velocity of cooling depends also on the temperature of the body; we may therefore assume that

$$F(t+\theta) - F(\theta) = \phi(t)\, a^{\theta};$$

$$\therefore\ \phi(t) = \frac{F'(\theta)}{a^{\theta}}\, t + \frac{F''(\theta)}{a^{\theta}} \cdot \frac{t^2}{1.2} + \&c.$$

The first side of which equation is independent of θ; and the last side will be so, if $\frac{F^r(\theta)}{a^{\theta}} = C_r$, a constant for all values of r.

This gives $F'(\theta) = C_1 a^{\theta}$,

$$F''(\theta) = C_1 a^{\theta} \log a = C_2 a^{\theta};$$

$$\therefore\ C_2 = C_1 \log a,$$

$$C_3 = C_1 (\log a)^2;$$

$$\ldots\ldots = \ldots\ldots$$

and $F\theta = \frac{C_1}{\log a} \cdot a^{\theta} = m \cdot \theta$; suppose

$$\therefore\ \phi t = C_1 \left\{ t + \log a \cdot \frac{t^2}{1.2} + (\log a)^2 \frac{t^3}{1.2.3} + \ldots \right\},$$

$$= m(a^t - 1),$$

and $V = m a^{\theta} (a^t - 1)$.

41. To determine a for the glass thermometer.

By the table (Art. 39), it appears, that when θ increases by 20^0, t remaining the same, the velocity of cooling is increased in the ratio 1.165 to 1.

If $\therefore\ V_0 = m a^0 (a^t - 1)$,

$$V_{20} = m a^{20} (a^t - 1) = V \cdot 1.165;$$

$$\therefore\ 1.165 = a^{20},$$

$$a = \sqrt[20]{1.165} = 1.0077 \text{ nearly.}$$

Cor. Similarly we shall find $m = 2.037$.

42. Had we started in this investigation of the form of the function which expresses the law of cooling, by assuming that if we commence our measure of temperature from a certain point, at which, under the particular circumstances, no radiation takes place, the amount of radiation in *space* will vary in a geometric ratio with the temperature; and conceived that the same hypothesis would require us to adopt a similar ratio for the velocity of reception, we should have

$$\text{velocity of radiation} = a^{t+\theta+\alpha},$$

$$\text{.....reception} = a^{\theta+\alpha};$$

$$\text{hence} \quad V = a^{\alpha+\theta}\,(a^t - 1),$$

$$= m a^{\theta}\,(a^t - 1),$$

which is the same law as before.

This latter mode of viewing the subject is more consistent with an undulatory hypothesis.

43. By making similar observations with a silvered thermometer, it appears that the velocity in cooling is represented by the same formula with the exception that m is considerably larger; a remains precisely the same.

44. *On cooling in gases.* If we find the velocities of cooling in a gas, and subtract from them respectively, the corresponding difference for the same excess of temperature in vacuo, we obtain the partial velocities of cooling due to the contact of the gas.

By means of a great number of observations MM. Dulong and Petit, arrived at the following laws.

1. The loss of heat due to the contact of a gas is *cæteris paribus*, independent of the nature of the cooling surface.

2. The loss depends both on the density and the temperature of the gas; but the dependence is such, that the loss is constant if the elasticity remains constant.

3. The losses of heat due to the contact of a gas, increase with the excess of temperature, following a law which does not involve the elasticity with the temperature.

4. The law of loss for the same temperature due to a difference of elasticity is

$$\frac{P}{P'} = \left(\frac{p}{p'}\right)^c,$$

c being different for different gases, p and p' the elasticities, P, P' the corresponding cooling powers.

5. The law of cooling of a gas is independent of the nature and density of the gas, that is, the cooling due to the contact of one gas under a given pressure, is to that due to the contact of another under the same pressure, in a ratio independent of the excess of temperature.

6. The law of the velocity of cooling due to the contact of a gas, is

$$V = np^c . t^{1.233},$$

where n varies with the extent of the cooling surface and the nature of the gas.

From all which it appears that the velocity of cooling in air is expressed by the formula

$$u = ma^\theta (a^t - 1) + np^c t^{1.233},$$

when $a = 1.0077$.

45. To compare the velocities of cooling of different surfaces presented under the same circumstances to the same gas.

By the first law, n will not vary with the nature of the surface, whilst m increases very rapidly as the radiating power increases.

Let therefore V' be the velocity of cooling for a more intensely radiating surface than glass;

$$\therefore \frac{V}{V'} = \frac{m a^\theta (a^t - 1) + n p^c t^{1 \cdot 233}}{m' a^\theta (a^t - 1) + n p^c t^{1 \cdot 233}},$$

$$= \frac{M t^{-\cdot 233} \left(\frac{a^t - 1}{t}\right) + 1}{M' t^{-\cdot 233} \left(\frac{a^t - 1}{t}\right) + 1}.$$

If $t = 0$, we get $\frac{V}{V'} = \frac{M}{M'}$,

if $t = \infty$, it is also $\frac{M}{M'}$,

if t be small $\frac{V}{V'} = \frac{M t^{-\cdot 233} \log a + 1}{M' t^{-\cdot 233} \log a + 1}$,

which evidently diminishes as t increases.

The numerator of the differential coefficient of the fraction with respect to t is

$$\left\{M' (a^t - 1) + t^{1 \cdot 233}\right\} (M a^t \log a + 1.233\, t^{\cdot 233})$$
$$- \left\{M (a^t - 1) + t^{1 \cdot 233}\right\} (M' a^t \log a + 1.233\, t^{\cdot 233})$$
$$= 1.233\, (M' - M)\, t^{\cdot 233} (a^t - 1) - (M' - M) \log a\, t^{1 \cdot 233} a^t,$$

which will vanish when

$$1.233\, (a^t - 1) = \log a \,.\, t,$$

let t_1 be the value of t from this equation, and it is obvious that when t is smaller than t_1, the differential coefficient is

negative, and when greater it is positive, hence $\frac{V}{V'}$ diminishes as t increases from 0 to t_1, and increases as $\frac{V}{V'}$ increases from t_1 to ∞.

Thus, it appears, that for a small excess of temperature, the velocity of cooling of a body of small radiating power increases in a faster ratio than that of a body of high radiating power, provided the excess of temperature is small; but that the converse is the case when the excess is large.

46. The remainder of our sketch of radiant heat must be very brief.

Some of the leading facts are the following:

The time of its passage through a given space is inappreciable.

The nature of the emitting substance, and the degree of polish of its surface, affect the radiation: the higher the degree of polish the less the radiation.

The degree of radiation is influenced by the colour of the surface, and is most intense for black and least for white. The experiments in this case are not very satisfactory, owing to the *nature* of the substance changing with its colour.

The absorptive power of a surface is in the same proportion as the radiative, and the reflective power in the inverse ratio.

The reflective power of most non-metallic substances is very small, whatever be their degree of polish.

47. Radiant heat appears to be either of two kinds dissimilar the one from the other, or which is more probable,

owing its mode of propagation through a substance to different circumstances.

When a substance gives heat without light, the heat is very different from that which accompanies light: for it will not pass *directly* through a glass screen; it will not affect *black* surfaces as the other does, provided they are not the boundaries of otherwise absorptive media. Although it does not pass through glass, it passes freely through rock salt, and is refracted. It may be added that a *very small* radiation is effected even through glass.

CHAPTER III.

CONDUCTION.

48. HAVING stated and examined the principles which regulate the emission of heat from the surfaces of bodies, it becomes necessary in the next place to examine the mode by which the heat is transferred from one part of the body to another. When speaking of solids, this is termed *conduction*. It is clear from the term itself that we do not include under this head the transfer of heat by radiation across a substance, such as we before mentioned in the case of rock salt, which suffers 92 out of every hundred incident rays to pass through it at a certain thickness. The latter mode of transfer is simple radiation. Nor do we include that transfer which takes place amongst the particles themselves, carrying with them the caloric they have acquired, which we designated *convection*.

The subject of conduction is confined to the transfer of heat by communication between the adjacent particles of a body.

49. The degree of readiness with which heat is communicated from particle to particle differs greatly in different bodies. We can hold a glass rod with impunity in the hand, when the extremity of it an inch from the fingers is in the flame, whilst if we attempted the same with a metal rod it would burn the fingers.

The degree of penetrability to heat, or the interior conductivity of a substance, will evidently be properly measured by the quantity of heat which in an unit of

time passes across a plane area whose extent is an unit; a surface parallel to this, on one side being retained at the temperature 0, and a surface on the other side at the temperature 1, no heat being lost laterally. The temperature 1 we shall in general suppose 100^0 of centigrade.

50. Another constant which we shall require in our investigations, is the exterior conductivity or the velocity of cooling.

We have already stated that its measure is the quantity of heat which in an unit of time escapes from an unit of surface retained constantly at the temperature 1 into a medium at the temperature 0: call it h.

It is only necessary to add that if the latter medium have not the temperature 0, but a constant temperature s, the conductivity of the body relative to this medium is $h(1-s)$.

We proceed now to the mathematical solution of the problem of conduction, the object of which is to determine the constant or variable state of heat in different parts of a body, subject to the action of foci of external heat. The mere results of computation taken apart from this verification by experiment, offer an interesting branch of pure analysis. It is not, however, consistent with the design of the present work to enter largely into the solution of Differential Equations. All that admits of being tested by experiment, is easily obtained by very simple analysis, and however interesting may be the more general details to the pure mathematician, they promise to offer little, save a ponderous drag to the progress of Physical Theory.

The speculations of M. Poisson however are admirable in themselves, and my best apology for passing them by, is that few would be inclined to enter with me into such extremely *general* investigations, especially when they had learnt that the results could rarely be offered in a shape

capable of being compared with experiment without the introduction of an hypothesis, reducing them in fact, to the simple condition in which they are usually presented.

The hypothesis, too, of universal radiation, whereby each particle communicates a portion of its heat to all other particles at the same time that it receives a portion of heat from them, admirably as it appears to account for many phenomena, especially some very striking ones, as the radiation of cold, the law of cooling of Dulong and Petit and the like, is an hypothesis so extremely complex in itself, that even were it the known law of nature, we should be compelled at once to renounce the idea by which it has usually been presented to our minds. We could not as we do now, conceive caloric a fluid traversing space and flowing amongst the particles of bodies. It would be very hard to imagine any thing like this, and thus we are dispossessed of a notion which appears so probable that its loss is not compensated for, unless it appear that it is incapable of comprehending all the phenomena.

Now I consider this is by no means proved, on the contrary there appears to me as much probability for the one hypothesis as for the other in these cases. Thus, for Dulong's law, if E be the elastic force of the *heat* in the body, D its density, E' the elastic force of the external heat, the velocity with which particles leave the surface

$$= c \cdot \frac{E - E'}{D};$$

and hence the quantity radiated in an unit of time

$$= c \cdot (E - E').$$

Now that E and E' should vary as some quantity raised to the power of the temperature is very probable, but it does not obviously result from the one hypothesis more than from the other.

On the uniform rectilinear motion of heat.

51. PROB. A homogeneous solid is bounded by two parallel planes; the lower being retained constantly at the temperature a, and the upper at the temperature b; to find the temperature at any intermediate point; the exterior conductivity not being considered.

Let c be the thickness of the solid, the area of its section $= 1$; and suppose it divided into slices by planes parallel to the bounding planes: let these slices be equal, and conceive each collected in a plane which is parallel to its boundaries and midway between them. Then we may suppose the temperature of each of these collected masses uniform throughout it.

Let x, $x + \delta x$ be the distance of two such masses from the lower plane of the solid;

$$v,\ v + \delta v \text{ their temperatures;}$$

then the quantity of heat which the lower transmits in a given time to the upper is proportional to δv (35);

$$\text{or if } v = A + Bx + Cx^2 + \ldots$$

$$\text{it is proportional to } B\delta x + 2Cx\delta x + \ldots$$

but the same quantity of heat must pass in the same time from any one of these masses to the next, so that this quantity must be independent of x, and $\therefore$ C, D, &c. all equal zero;

$$\therefore\ v = A + Bx,$$

$$a = A \text{ when } x = 0,$$

$$b = A + Bc \ldots x = c,$$

$$v = a - \frac{a - b}{c}x \text{ is the temperature required.}$$

Cor. 1. The quantity of heat transmitted across a section is proportional to B and is equal to KB

$$= K \cdot \frac{a-b}{c}.$$

In the particular case where $a = 1$, $b = 0$, $c = 1$, it is equal to K.

K is the measure of the interior conductivity of the substance.

Cor. 2. In the case in which the slices are taken at indefinitely small distances from each other

$$b = a + \frac{da}{dx}\delta x + \ldots$$

$$c = \delta x;$$

therefore the quantity of heat transmitted across an indefinitely thin plane, or across any section is $-K\frac{da}{dx}$; a being the temperature at that section. This is called the *flux of heat.*

Cor. 3. It is not requisite for this latter Corollary that the exterior conductivity should be zero.

52. Prob. Instead of supposing the upper surface of the solid retained at the temperature b, let us suppose the upper surface to be terminated in the air whose temperature is β; to find the temperature at any point.

Let the notation of the last problem be retained, and suppose the temperature of the upper surface $= b$ as before, then

$$v = a - \frac{a-b}{c} \cdot x,$$

and it remains to determine b.

Let k be the exterior conductivity of the upper surface,

K the interior,

then since the same quantity of heat must pass across any particular plane, as passes from the upper plane to the air, we must have

$$K \cdot \frac{a-b}{c} = k(b-\beta);$$

$$\therefore \quad b = \frac{Ka + kc\beta}{K + kc},$$

$$\text{and} \quad a - b = \frac{kc(a-\beta)}{K+kc};$$

$$\therefore \quad v = a - \frac{k(a-\beta)}{K+kc} \cdot x.$$

53. Prob. A rectangular prismatic bar, whose section is a square, and whose length is infinite, has every point of its extremity exposed to the action of a constant source of heat; it is required to find the law of distribution, when it has become permanent.

Let l be the *side* of the section,

K the interior and h the exterior conductivity,

then it is evident, that if we take any portion whose length is δx, the quantity of heat which escapes from its surface must be the difference between the quantities which traverses the lower and upper plane which bound it.

But if F be the flux of heat across the lower section, whose distance from the source of heat is x, we have by (51),

$$F = -K\frac{dv}{dx}l^2,$$

and the quantity which traverses the upper section is

$$F + \frac{dF}{dx}\delta x + \ldots = -K\frac{l^2 dv}{dx} - K\frac{l^2 d^2 v}{dx^2}\delta x - \ldots$$

and the excess of the former of these above the latter is

$$+K\frac{l^2d^2v}{dx^2}\delta x+\ldots.$$

Also, the quantity emitted by the surface of this portion, supposing it uniformly of the temperature v is $4hlv\delta x$,

hence, equating these quantities and taking the limit, we obtain

$$Kl\frac{d^2v}{dx^2}=4hv,$$

the integral of this equation is

$$v=Ae^{cx}+Be^{-cx} \text{ where } c^2=\frac{4h}{Kl};$$

but when x is a, $v=0$;

and when $x=0$, $v=a$;

hence $0=Ae^{a}$, $\therefore A=0$;

$$a=B;$$

$$\therefore v=ae^{-cx},$$

represents the permanent state of temperature.

Cor. If two bars of different size be constructed of the same metal, and x_1 x_2 be the distances from the origin of heat, at which they acquire the same temperature, and l_1 l_2 be the dimensions of a side of their section respectively,

$$ae^{-2\sqrt{\frac{h}{Kl_1}}x_1}=ae^{-2\sqrt{\frac{h}{Kl_2}}x_2};$$

$$\therefore \frac{x_1}{\sqrt{l_1}}=\frac{x_2}{\sqrt{l_2}} \quad \text{or} \quad \frac{x_1^2}{x_2^2}=\frac{l_1}{l_2};$$

or the distances are as the square roots of the linear dimensions of the prisms; which agrees with experiment.

54. Prob. To find the motion of heat in the ring.

The figure whose state of heat we are about to consider at present, is that generated by any plane curve revolving about an axis in its own plane, but at a considerable distance from the curve itself, which we consider a close one.

The magnitude of the curve is supposed so small, that the temperature of all points in it is not sensibly different. The path traced out by the center of gravity of the revolving area is a circle of such diameter, that the product of the revolving area into the arc of this circle, is the volume of the solid ring comprised between the limits of the arc. Now, if the figure be, as we suppose it is, a symmetric curve, such as a circle, an ellipse, or a square, the center of gravity of the arc coincides with that of the area, hence also the *surface* of the portion of the ring comprised between the same limits will be equal to the product of the arc traced out by the center of gravity, into the length of the periphery of the revolving figure. We assume Newton's law, that the quantity of heat which escapes by the surface into a vacuum, varies *cæteris paribus* as the temperature. We shall adopt Dulong's law in a subsequent solution.

Taking any position of the center of gravity as origin, let x be measured along the path described by it, so that x will be the arc of a circle whose radius is the distance r of the center of gravity of the figure from the axis.

Let S be the area of the curve,

K the conductivity of the substance;

hence, $-KS\frac{dv}{dx}dt$ represents the quantity which passes across a section at distance x from the origin, and whose area is S, and the quantity which is acquired by a slice is

$$+KS.\frac{d^2v}{dx^2}\delta x\delta t,$$

being the difference at points x and $x+\delta x$.

But if h be the measure of the exterior conductivity; l the periphery,

$$hl\delta x . v\delta t \text{ is the quantity that escapes.}$$

$\therefore$ $KS\frac{d^2v}{dx^2}\delta x\delta t - hlv\delta x\delta t$ is the quantity that passes into δx in the time δt more than escapes.

Let C be the specific heat of the substance.

$$SD\delta x \text{ is the mass of the slice;}$$

$$\therefore\ CSD\delta x \text{ is its specific heat;}$$

$\therefore$ $CSD\delta x\delta v$ is the heat requisite to raise the slice by the temperature δv.

But the heat acquired is as above; and by hypothesis, the temperature is raised δv.

$$\therefore\ CSD\delta x\delta v = KS\frac{d^2v}{dx^2}\delta x\delta t - hlv\,\delta x\delta t,$$

$$\frac{dv}{dt} = \frac{K}{CD}\cdot\frac{d^2v}{dx^2} - \frac{hl}{CSD}\cdot v.$$

55. Prob. To find the state of equilibrium of heat in any body.

Let us take a small parallelopiped of the body any where in the mass bounded by planes parallel to the co-ordinate planes.

Let $\delta x\delta y\delta x$ be its magnitude;

then $-K\frac{dv}{dx}\delta x\delta y =$ the flux of heat across its base in an unit of time,

and the opposite face parts with a quantity

$$= -K\frac{dv}{dx}\delta x\delta y - K\frac{d^2v}{dx^2}\delta x\delta y\delta x;$$

$\therefore$ from the lower surface the parellelopiped acquires $K\frac{d^2v}{dz^2}\delta x\delta y\delta z$ more than it loses. In the same manner it acquires from the plane parallel to xz, $K\frac{d^2v}{dy^2}\delta x\delta y\delta z$ more than it loses, and from yz, $K\frac{d^2v}{dx^2}\delta x\delta y\delta z$, so that it gains altogether.

$$K\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)\delta x\delta y\delta z,$$

which in the case of permanent temperature must $=0$;

$$\therefore \frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}=0,$$

gives that state.

Cor. If K be not independent of the position of the point, as in the case of a body not homogenous and the like,

$\left(-K\frac{dv}{dz}\delta x\delta y\right)$ is the flux of heat across the base;

$\therefore \left\{-K\frac{dv}{dz}\delta x\delta y-\frac{d}{dz}\left(K\frac{dv}{dz}\right)\delta x\delta y\delta z\right\}$ is that across the upper surface,

hence, the quantity of heat acquired more than that lost by these surfaces is $\frac{d}{dz}.K\frac{dv}{dz}$;

and the equation for the equilibrium of temperature becomes

$$\frac{d}{dx}\left(K\frac{dv}{dx}\right)+\frac{d}{dy}\left(K\frac{dv}{dy}\right)+\frac{d}{dz}\left(K\frac{dv}{dz}\right)=0.$$

56. Prob. To find the variable state of temperature in any solid.

The mass of an elementary portion is $D\delta x\delta y\delta z$, and if C be the specific heat,

$C\frac{dv}{dt}dt$ is the quantity of heat acquired by an unit;

$\therefore$ the quantity which the element acquires is

$$CD\frac{dv}{dt}dt\delta x\delta y\delta z,$$

and it is also by last proposition,

$$K\delta t\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)\delta x\delta y\delta z;$$

$$\therefore \frac{dv}{dt}=\frac{K}{CD}\cdot\left(\frac{d^2v}{dx^2}+\frac{d^2v}{dy^2}+\frac{d^2v}{dz^2}\right)$$

Cor. If x, y, z the co-ordinates of the point under consideration, by dilatation or otherwise change with the time, $\frac{dv}{dt}$ the total differential will be replaced by

$$\frac{dv}{dt}+\frac{dv}{dx}\frac{dx}{dt}+\frac{dv}{dy}\cdot\frac{dy}{dt}+\frac{dv}{dz}\frac{dz}{dt};$$

$$\therefore \frac{dv}{dt}+\frac{dv}{dx}\cdot\frac{dx}{dt}+\frac{dv}{dy}\cdot\frac{dy}{dt}+\frac{dv}{dz}\cdot\frac{dz}{dt}$$

$$=\frac{1}{CD}\cdot\left\{\frac{d.K\frac{dv}{dx}}{dx}+\frac{d.K\frac{dv}{dy}}{dy}+\frac{d.K\frac{dv}{dz}}{dz}\right\}.$$

57. Prob. To find the conditions relative to the surface.

Suppose a normal drawn at the point xyz, and let it be called n,

then $-K\frac{dv}{dn}$ is the flux across the exterior section in an unit of time, but it is also hv;

$$\therefore hv+K\frac{dv}{dn}=0;$$

$$\text{but } \frac{dv}{dn} = \frac{dv}{dx}\cdot\frac{dx}{dn} + \frac{dv}{dy}\cdot\frac{dy}{dn} + \frac{dv}{dz}\cdot\frac{dz}{dn},$$

$\frac{dx}{dn}$, &c. being the limits of the quotients of δx by δn....

hence, $\frac{dx}{dn} = \cos \angle$ made by normal with axis of x,

$$= \frac{\frac{du}{dx}}{\sqrt{\left(\frac{du}{dx}\right)^2 + \left(\frac{du}{dy}\right)^2 + \left(\frac{du}{dz}\right)^2}} = \frac{m}{\sqrt{m^2+n^2+p^2}} = \frac{m}{q},$$

suppose; $u = 0$ being the equation to the surface.

$\therefore\ m\frac{dv}{dx} + n\frac{dv}{dy} + p\frac{dv}{dz} + \frac{h}{K}vq = 0$ is the equation required.

58. Prob. To find the motion of heat in a sphere.

Let r be the radius of any spherical surface;

$$r^2 = x^2 + y^2 + z^2;$$

$$\text{then } \frac{dv}{dx} = \frac{dv}{dr}\cdot\frac{dr}{dx} = \frac{dv}{dr}\cdot\frac{x}{r};$$

$$\frac{d^2v}{dx^2} = \frac{x^2}{r^2}\frac{d^2v}{dr^2} + \frac{dv}{dr}\left(\frac{1}{r} - \frac{x^2}{r^3}\right),$$

$$\frac{d^2v}{dy^2} = \frac{y^2}{r^2}\frac{d^2v}{dr^2} + \frac{dv}{dr}\left(\frac{1}{r} - \frac{y^2}{r^3}\right),$$

$$\frac{d^2v}{dz^2} = \frac{z^2}{r^2}\frac{d^2v}{dr^2} + \frac{dv}{dr}\left(\frac{1}{r} - \frac{z^2}{r^3}\right);$$

$$\therefore\ \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = \frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr};$$

and the equation of motion becomes

$$\frac{dv}{dt} = \frac{K}{CD} \cdot \left(\frac{d^2v}{dr^2} + \frac{2}{r} \frac{dv}{dr} \right).$$

Cor. It is obvious that the equation relative to the surface is $hv + K \frac{dv}{dr} = 0$.

Integration of the equations which represent the permanent state of heat in bodies.

59. The equation which represents the permanent state in the ring is $\frac{d^2v}{dx^2} = \frac{hl}{KS} v$; $\frac{dv}{dt}$ being in this case equal to zero.

It is obvious that the solution of this equation is

$$v = Ae^{-x\sqrt{\frac{hl}{KS}}} + Be^{x\sqrt{\frac{hl}{KS}}}.$$

Let $v_1 v_2 \ldots$ be the temperatures of points whose distances from the origin (measured along the circumference) are $x_1 x_2 \ldots\ldots$

then $v_1 = Ae^{-x_1 \alpha} + Be^{x_1 \alpha}$,

denoting $\sqrt{\frac{hl}{KS}}$ by α,

$$v_2 = Ae^{-x_2 \alpha} + Be^{x_2 \alpha},$$

$$\ldots = \ldots\ldots$$

Suppose the distances equal, so that

$$v_2 - v_1 = v_3 - v_2 = \lambda;$$

$$\therefore\ v_1 + v_3 = Ae^{-ax_1} + Be^{ax_1} + Ae^{-a(x+2\lambda)} + Be^{a(x_1+2\lambda)}$$

$$= Ae^{-a(x_2-\lambda)} + Be^{a(x_2-\lambda)} + Ae^{-a(x_2+\lambda)} + Be^{a(x_2+\lambda)}$$

$$= Ae^{-ax_2}(e^{a\lambda} + e^{-a\lambda}) + Be^{ax_2}(e^{a\lambda} + e^{-a\lambda})$$

$$= (e^{a\lambda} + e^{-a\lambda})(Ae^{-ax_2} + Be^{ax_2})$$

$$= (e^{a\lambda} + e^{-a\lambda})\, v_2$$

$$= qv_2 \text{ suppose;}$$

$$\therefore\ v_3 = qv_2 - v_1,$$

$$v_4 = qv_3 - v_2,$$

$$\ldots = \ldots\ldots\ldots$$

Whence it follows that if the surface be divided into any number of equal parts, the temperatures at the points of division form a recurring series, whose scale of relation is $q - 1$.

60. These results have been abundantly confirmed by experiments as varied as possible. It will suffice to give the results of one.

Three thermometers were placed at different points at the distance of $\frac{1}{8}$ of the circumference from each other in a ring unequally heated, so that the temperatures marked by them were 66^0, $50\frac{7}{12}{}^0$, 44^0 respectively, whilst that of the air was $17\frac{2}{3}{}^0$, thus giving for v_1, v_2, v_3 the values $48\frac{1}{3}$, $32\frac{11}{12}$, $26\frac{1}{3}$ respectively.

With these observations the calculated value of q is 2.268.

In order to verify this, another thermometer was placed at the same distance from the third thermometer, that the first is from it, thus, making v_1, v_3, v_4 temperatures at distances of a quarter of the circumference from each other.

Now from the value of v_4 it appeared that

$$\frac{v_1 + v_4}{v_3} = 3.140 = r \text{ suppose;}$$

$$\therefore q = e^{\alpha\frac{\pi}{4}} + e^{-\alpha\frac{\pi}{4}},$$

$$r = e^{\alpha\frac{\pi}{2}} + e^{-\alpha\frac{\pi}{2}}$$

$$= q^2 - 2 = 3.143;$$

a result which completely verifies the accuracy of the formula in this case.

61. PROB. To find the permanent state of the heat in a rectangular solid of indefinite length.

Suppose it placed in a medium whose temperature is zero, and every point of the surface except the base retained at that temperature. Conceive every point of the plane of yz to be retained at the constant temperature 1, the axis of x being in the direction of the latter; and suppose further, that there is no transmission parallel to z, so that $\frac{d^2v}{dz^2}$ may $= 0$.

Let the solid be symmetrical about the axis of x, then it is evident that we must satisfy the following conditions, viz. make

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = 0;$$

$v = 0$ at every point of the surface,

and $v = 1$ at every point in the plane of yz.

Let $2l$ be the breadth perpendicularly, then if $-l$ or $+l$ be put for y; v must $= 0$ whatever be x; and when $x = 0$, every value of y between $-l$ and $+l$ must render $v = 1$; and finally, when x is large v must be small.

A particular integral of the equation is evidently

$$v = e^{\mp nx} \cos ny.$$

Now the condition that when x is large the value of v shall be small, renders it necessary to confine ourselves to the solution $v = e^{-nx} \cos ny$; the further condition that when $y = \pm l$, v must equal zero for all the values of x, gives $\cos nl = 0$;

$$\therefore\ nl = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \ldots\ldots \frac{2m+1}{2} \cdot \pi;$$

$$\text{hence } v = e^{-\frac{2m+1}{2l}\pi x} \cos \frac{2m+1}{2l} \cdot \pi y.$$

In order to satisfy the other condition, that when $x = 0$, $v = 1$ for all values of y less than $\pm l$; it is requisite to take the complete value of v, viz.

$$v = a_1 e^{-\frac{\pi x}{2l}} \cos \frac{\pi y}{2l} + a_2 e^{-\frac{3\pi x}{2l}} \cos \frac{3\pi y}{2l} + a_3 e^{-\frac{5\pi x}{2l}} \cos \frac{5\pi y}{2l} + \ldots.$$

and to determine $a_1 a_2 \ldots\ldots$ so that the condition

$$1 = a_1 \cos \frac{\pi}{2l} y + a_2 \cos \frac{3\pi}{2l} y + \ldots\ldots$$

may be satisfied for all values of y between $-l$ and $+l$.

To find the values of a_1, a_2.....

$$\because\ 1 = a_1 \cos \frac{\pi}{2l} y + a_2 \cos \frac{3\pi}{2l} y + \ldots\ldots a_r \cos Ky + \ldots\ldots$$

multiply by $\cos Ky$ and integrate;

$$\therefore\ \frac{1}{K} \sin Ky = \ldots\ldots + a_s \int \cos Sy \cos Ky\, dy + a_r \int \cos^2 Ky\, dy + \ldots$$

$$= \ldots\ldots + \frac{a_s}{2} \left\{ \frac{\sin (S-K)y}{S-K} + \frac{\sin (S+K)y}{S+K} \right\}$$

$$+\frac{a_r}{2}\left(y+\frac{\sin 2Ky}{2K}\right)+\ldots\ldots$$

which taken between the limits $y=0$, $y=l$, gives

$$\pm\frac{1}{K}=+\frac{a_r l}{2},$$

according as K is of the form $4m+1$ or $4m+3$.

$$\text{hence}\quad a_r=\pm\frac{2}{Kl}=\pm\frac{4}{(2r-1)\pi};$$

$$\therefore\ 1=\frac{4}{\pi}\cdot\left(\cos\frac{\pi}{2l}y-\frac{1}{3}\cos\frac{3\pi}{2l}y+\frac{1}{5}\cos\frac{5\pi}{2l}y-\ldots.\right);$$

$$\text{or}\quad\frac{\pi}{4}=\cos\frac{\pi y}{2l}-\frac{1}{3}\cos\frac{3\pi}{2l}y+\ldots\ldots$$

$$\text{and}\quad v=\frac{4}{\pi}\cdot\left(e^{-\frac{\pi x}{2l}}\cos\frac{\pi y}{2l}-\frac{1}{3}e^{-\frac{3\pi x}{2l}}\cos\frac{3\pi y}{2l}+\ldots\ldots\right)$$

is the value of the temperature required.

62. Prob. To find the quantity of heat which in an unit of time traverses across a plane parallel to that of yz, at a distance x from the origin.

This quantity is represented by $-K\frac{dv}{dx}$ (51), the area of the plane being unity;

hence the part of it at the point y across an element is

$$+\frac{2K}{l}\cdot\left(e^{-\frac{\pi x}{2l}}\cos\frac{\pi y}{2l}-e^{-\frac{3\pi}{2l}}\cos\frac{3\pi y}{2l}+\ldots\ldots\right)\delta yZ,$$

Z being the breadth parallel to z;

and the whole quantity is the integral of this from

$$y=+l,\ y=-l.$$

$$\text{Now } \frac{2K}{l}\int\left(e^{-\frac{\pi x}{2l}}\cos\frac{\pi y}{2l} - \ldots\ldots\right)Z dy,$$

$$= \frac{4KZ}{\pi}\left(e^{-\frac{\pi x}{2l}}\sin\frac{\pi y}{2l} - \frac{1}{3}e^{-\frac{3\pi x}{2l}}\sin\frac{3\pi y}{2l} + \ldots\ldots\right);$$

and between the limits it becomes

$$\frac{8KZ}{\pi}\left(e^{-\frac{\pi x}{2l}} + \frac{1}{3}e^{-\frac{3\pi x}{2l}} + \frac{1}{5}e^{-\frac{5\pi x}{2l}} + \ldots\ldots\right),$$

the quantity required.

63. Prob. To find the quantity of heat which traverses a plane parallel to xz;

$$-K\frac{dv}{dy} = \frac{2K}{l}\left(e^{-\frac{\pi x}{2l}}\sin\frac{\pi y}{2l} - e^{-\frac{3\pi x}{2l}}\sin\frac{3\pi y}{2l} + \ldots\ldots\right);$$

$$\therefore Z\int\left(-K\frac{dv}{dy}\right)dx = C - \frac{4K}{\pi}\left(e^{-\frac{\pi x}{2l}}\sin\frac{\pi y}{2l} - e^{-\frac{3\pi x}{2l}}\sin\frac{3\pi y}{2l} + \ldots\right)Z;$$

this taken from $x = x$ to $x = \infty$ is

$$\frac{4KZ}{\pi}\left(e^{-\frac{\pi x}{2l}}\sin\frac{\pi y}{2l} - \frac{e^{-\frac{3\pi x}{2l}}}{3}\sin\frac{2\pi y}{2l} + \ldots\ldots\right).$$

If we put $y = l$ in this expression, we obtain the quantity which escapes by the surface on one side of x, between the point x, and the extremity of the bar; it is

$$\frac{4KZ}{\pi}\left(e^{-\frac{\pi x}{2l}} + \frac{1}{3}e^{-\frac{3\pi x}{2l}} + \frac{1}{5}e^{-\frac{5\pi x}{2l}} + \ldots\ldots\right);$$

and it is evident that the same quantity will escape from the other surface parallel to x, between the same limits, hence, the whole quantity escaping by the surface which lies beyond a plane perpendicular to x, at a distance x from the origin, is

$$\frac{8KZ}{\pi}\left(e^{-\frac{\pi x}{2l}}+\frac{1}{3}e^{-\frac{3\pi x}{2l}}+\ldots\ldots\right);$$

but, by the last article, this is also the quantity which traverses the same plane, a result which is of course requisite.

64. Suppose a rectangular bar, as in the last proposition, so acted on, that the temperature of the base is a function $f(y)$ of its distance from the extremity B, whilst every point of the planes parallel to yz which terminate the solid, is retained at the constant temperature zero.

Our object now is to obtain the values of the assumed quantities in the series

$$v = a_1 e^{-\frac{\pi x}{2l}} \sin\frac{\pi y}{2l} + a_2 e^{-\frac{2\pi x}{2l}} \sin\frac{2\pi y}{2l} + a_3 e^{-\frac{3\pi x}{2l}} \sin\frac{3\pi y}{2l} + \ldots\ldots$$

the sine being substituted for the cosine, in order to make the function vanish when $y = 0$, the origin being taken in the surface. It would require that both be retained in some cases.

The condition to be satisfied is that when $x = 0$, $v = f(y)$ between the limits $y = 0$, $y = 2l$.

$$\therefore\ f(y) = a_1 \sin\frac{\pi y}{2l} + a_2 \sin\frac{2\pi y}{2l} + \ldots\ldots a_r \sin\frac{r\pi y}{2l} + \ldots\ldots$$

$$\therefore\ \int \sin\frac{r\pi y}{2l} f(y)\,dy = \ldots + a_r \int \frac{1+\cos\frac{r\pi y}{l}}{2}\,dy + \ldots$$

$$\text{or}\quad \int_0^{2l} \sin\frac{r\pi y}{2l} f(y)\,dy = a_r . l;$$

hence we obtain

$$lv = e^{-\frac{\pi x}{2l}} \sin\frac{\pi y}{2l} \int_0^{2l} \sin\frac{\pi y}{2l} f y\,dy$$

$$+ e^{-\frac{2\pi x}{2l}} \sin\frac{2\pi y}{2l} \int_0^{2l} \sin\frac{2\pi y}{2l} f y\,dy + \ldots\ldots$$

which may be written

$$lv = \int_0^{2l} f a\, da \left(e^{-\frac{\pi x}{2l}} \sin\frac{\pi y}{2l} \sin\frac{\pi a}{2l} \right.$$

$$\left. + e^{-\frac{2\pi x}{2l}} \sin\frac{2\pi y}{2l} \sin\frac{2\pi a}{2l} + \ldots\ldots \right),$$

$$= \Sigma_{n0}^{\infty} \int_0^{2l} f a\, da \,.\, e^{-\frac{n\pi x}{2l}} \sin\frac{n\pi y}{2l} \sin\frac{n\pi a}{2l}.$$

65. Cor. The above equation is susceptible of another very convenient form.

$$\because \sin\frac{n\pi y}{2l} \sin\frac{n\pi a}{2l} = \frac{1}{2}\left\{\cos\frac{n\pi}{2l}(y-a) - \cos\frac{n\pi}{2l}(y+a)\right\},$$

$$\text{if } \Sigma_{n0}^{\infty}\, e^{-\frac{n\pi x}{2l}} \cos\frac{n\pi}{2l}(y-a) = F(x,\ y-a),$$

$$2lv = \int_0^{2l} f a\, da \left\{F(x,\ \overline{y-a}) - F(x,\ \overline{y+a})\right\}$$

To sum the series $\Sigma_n e^{-nq} \,.\, \cos nr = u$; it is evident that

$$2u = \left\{e^{-q}\left(e^{r\sqrt{1}} + e^{-r\sqrt{-1}}\right) + e^{-2q}\left(e^{2r\sqrt{1}} + e^{-2r\sqrt{1}}\right) + \ldots\ldots\right\}$$

$$= e^{-q+r\sqrt{1}}\left(1 + e^{-q+r\sqrt{1}} + e^{-2q+2r\sqrt{1}} + \ldots\ldots\right)$$

$$+ e^{-q-r\sqrt{1}}\left(1 + e^{-q-r\sqrt{1}} + \ldots\ldots\right)$$

$$= e^{-q}\frac{e^{r\sqrt{1}}}{1 - e^{-q+r\sqrt{1}}} + e^{-q}\frac{e^{-r\sqrt{1}}}{1 - e^{-q-r\sqrt{1}}}$$

$$= e^{-q}\left\{\frac{e^{r\sqrt{1}} + e^{-r\sqrt{-1}} - \left(e^{-q} + e^{-q}\right)}{1 - e^{-q}\left(e^{r\sqrt{1}} + e^{-r\sqrt{1}}\right) + e^{-2q}}\right\}$$

$$= e^{-q}\left\{\frac{2\cos r - 2e^{-q}}{1 - 2e^{-q}\cos r + e^{-2q}}\right\};$$

$$\therefore F(x, y-a) = e^{-\frac{\pi x}{2l}} \frac{\cos\frac{\pi}{2l}(y-a) - e^{-\frac{\pi x}{2l}}}{1 - 2e^{-\frac{\pi x}{2l}}\cos\frac{\pi}{2l}(y-a) + e^{-\frac{2\pi x}{2l}}};$$

$$\text{and } 2lv = \int_0^{2l} f a da \cdot e^{-\frac{\pi x}{2l}} \left\{ \frac{\cos\frac{\pi(y-a)}{2l} - e^{-\frac{\pi x}{2l}}}{1 - 2e^{-\frac{\pi x}{2l}}\cos\frac{\pi(y-a)}{2l} + e^{-\frac{2\pi x}{2l}}} \right.$$

$$\left. - \frac{\cos\frac{\pi(y+a)}{2l} - e^{-\frac{\pi x}{2l}}}{1 - 2e^{-\frac{\pi x}{2l}}\cos\frac{\pi(y+a)}{2l} + e^{-\frac{2\pi x}{2l}}} \right\};$$

which is the complete integral of the equation subject to its conditions, and is under a finite form.

66. PROB. To find the state of equilibrium of heat in a rectangular solid, one end of which is kept at an uniform temperature 1, and the rest exposed to a medium at temperature 0, but the surfaces not necessarily having that temperature.

We take as before the plane of yz as the heated part, and its middle point as the origin: the difference between this proposition and the former, consisting in that the section is *finite*, and consequently there will be a variation by the alteration of z.

Let $2l$ be the length of the section parallel to y;

$2r$ ..... z;

then we have to obtain such an integral of

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{d^2v}{dz^2} = 0,$$

as shall satisfy the conditions

$$hv + K\frac{dv}{dy} = 0, \text{ when } y = \pm l,$$

$$hv + K\frac{dv}{dx} = 0, \text{ when } x = \pm r;$$

$$v = 1, \text{ when } x = 0;$$

v diminishes as x increases.

The quantity e^{-nz} will evidently be a factor; and since the quantities y and x enter in a form similar to each other, if $\cos my$ be a factor of e^{-nz}, $\cos px$ must be another factor.

Assume therefore that $v = e^{-nz}\cos my \cos px$; and by substitution we obtain $n^2 = m^2 + p^2$.

The first condition gives

$$he^{-nz}\cos ml\cos px - mKe^{-nz}\sin ml\cos px = 0;$$

$$\therefore \tan ml = \frac{h}{mK}.$$

The second $\tan pr = \frac{h}{pK}$.

Let $ml = \theta$, $pr = \phi$;

$$\text{then } \tan\theta = \frac{hl}{K\theta}, \text{ or } \theta.\tan\theta = \frac{hl}{K};$$

$$\text{similarly, } \phi.\tan\phi = \frac{hr}{K};$$

and we must adopt all values of θ which give

$$\text{arc}\,\theta.\tan\theta \text{ a constant} = \frac{hl}{K};$$

and similarly of ϕ.

Now it is obvious that this will give us an ascending series of values of θ, going on from a finite quantity to infinity; and so of ϕ.

Let m_1, m_2 .....; p_1, p_2..... be the values of m and p in order. The general solution is then

$$v = a_1 e^{-n_1 x} \cos m_1 y \cos p_1 x + a_2 e^{-n_2 x} \cos m_1 y \cos p_2 x + \ldots$$

$$+ b_1 e^{-n_1 x} \cos m_2 y \cos p_1 x + \ldots$$

$$+ \ldots$$

and the further condition that $v = 1$ for $x = 0$ gives

$$1 = a_1 \cos m_1 y \cos p_1 x + a_2 \cos m_1 y \cos p_2 x + \ldots$$

$$+ b_1 \cos m_2 y \cos p_1 x + \ldots$$

for all values of y and x between $\pm l$, $\pm r$.

It is clear therefore that if we can reduce the expression to the form

$$1 = (a_1 \cos m_1 y + a_2 \cos m_2 y + \ldots)(\beta_1 \cos p_1 x + \beta_2 \cos p_2 x + \ldots);$$

we must separately have

$$1 = a_1 \cos m_1 y + a_2 \cos m_2 y + \ldots$$

$$1 = \beta_1 \cos p_1 x + \beta_2 \cos p_2 x + \ldots$$

This appears to be a solution having quite sufficient generality, and the values of $a_1 \ldots$ may be determined, whence the complete solution is obtained.

67. It will be perceived that the functions we here treat of are *discontinuous*.

The following is the general process of expressing a discontinuous function by means of a trigonometric series.

Let $f(x) = a_0 + a_1 \cos x + a_2 \cos 2x + \ldots$

$$+ b_1 \sin x + b_2 \sin 2x + \ldots$$

then $\int \cos Kx f x \, dx = \ldots + \int \frac{a_K}{2} (1 + \cos 2Kx) \, dx + \ldots$

$$A \int \sin (K + r) x \, dx + A' \int \cos (K + r) x \, . \, dx + \ldots$$

$$\therefore \int_0^{2\pi} \cos Kx f(x) \, dx = \pi a_K,$$

$$\text{and } \int_0^{2\pi} \sin Kx f(x) \, dx = \pi b_K,$$

$$\text{also } \int_0^{2\pi} f(x) \, dx = 2\pi a_0;$$

$$\therefore \pi f(x) = \tfrac{1}{2} \int_0^{2\pi} f x \, dx + \cos x \int_0^{2\pi} \cos x f x \, dx$$

$$+ \sin x \int_0^{2\pi} \sin x f x \, dx + \ldots$$

$$= \int_0^{2\pi} da \left\{ \tfrac{1}{2} + \cos a \cos x + \sin a \sin x \right.$$

$$\left. + \cos 2a \cos 2x + \sin 2a \sin 2x + \ldots \right\} fa.$$

It must be observed that the limits of these integrals are not to be taken at once, but the whole extent is to be broken up into the separate parts corresponding to the separate values of $f(a)$.

Another thing to be remarked, is, that in the integration we have supposed the whole series of changes to take place during the period of x changing from 0 to 2π. This necessarily restricts the theorem by introducing the hypothesis that the function is a reciprocating one, having the same value when $x = 2\pi$ as when $x = 0$.

No other restrictions are introduced. The period of x's change from 0 to 2π may be as extensive as we please by assuming another variable y depending on x; thus, suppose a function commences its value when the variable y is $= \theta$, and ends it when the value is ϕ: we must

assume x such a function of y, that when $y = \theta$ it may vanish, and when $y = \phi$ it may $= 2\pi$: this will be effected by supposing

$$x = \frac{2\pi}{\phi - \theta}(y - \theta),$$

and the limits coincide with those above.

In many cases $f(x)$ can be expressed by a series of sines or of cosines only: in such cases if

$$\phi(x) = a_0 + a_1 \sin x + a_2 \sin 2x + \ldots$$

$$\int_0^\pi \phi(x)\, dx = a_0 \pi,$$

$$\int_0^\pi \phi x \,.\, \sin x\, dx = \tfrac{1}{2} a_1 \pi,$$

$$\ldots\ldots\ldots = \ldots\ldots$$

$$\therefore \frac{\pi}{2}\phi(x) = \tfrac{1}{2}\int_0^\pi \phi x\, dx + \sin x \int_0^\pi \phi x \sin x\, dx + \ldots$$

and a similar expansion in terms of the cosines may be obtained.

68. Prob. To find a function such that it shall be $= c$ so long as x lies between 0, and a, and zero when x is between a and 2π.

Here $\int_0^{2\pi} \phi x\, dx = aC$;

$$\int_0^{2\pi} \phi(x) \sin x\, dx = (A - \cos x)\, C \text{ between } 0 \text{ and } a$$

$$= (1 - \cos a)\, C = C \operatorname{vers} a,$$

and zero beyond;

$$\int_0^{2\pi} \phi x \sin 2x\, dx = \frac{C}{2} \operatorname{vers} 2a,$$

$$\int_0^{2\pi} \phi x \sin 3x\, dx = \frac{C}{3} \operatorname{vers} 3a,$$

$$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$$

$$\int_0^{2\pi} \phi x \cos x\, dx = C \sin a\,;$$

$$\int_0^{2\pi} \phi x \cos 2x\, dx = \frac{C}{2} \sin 2a\,;$$

$$\ldots\ldots\ldots\ldots = \ldots\ldots\ldots$$

hence,

$$\frac{\pi \phi(x)}{C} = \tfrac{1}{2} a + \sin a \cos x + \tfrac{1}{2} \sin 2a \cos 2x + \tfrac{1}{3} \sin 3a \cos 3x + \ldots$$

$$+ \operatorname{vers} a \sin x + \tfrac{1}{2} \operatorname{vers} 2a \sin 2x + \ldots$$

It may be interesting to verify this result, especially as Fourier gives another form to the expression, which embraces only the second line of this.

Let $u = \sin\theta + \frac{1}{2}\sin 2\theta + \ldots$

$$= \sin(\pi - \theta) - \tfrac{1}{2}\sin 2(\pi - \theta) + \ldots$$

$$= \frac{1}{2\sqrt{-1}} \left\{ e^{(\pi-\theta)\sqrt{-1}} - \tfrac{1}{2} e^{2(\pi-\theta)\sqrt{-1}} + \ldots \right\}$$

$$- \frac{1}{2\sqrt{-1}} \left\{ e^{-(\pi-\theta)\sqrt{-1}} - \ldots \right\}$$

$$= \frac{1}{2\sqrt{-1}} \log \frac{1 + e^{(\pi-\theta)\sqrt{-1}}}{1 + e^{-(\pi-\theta)\sqrt{-1}}} = \frac{1}{2\sqrt{-1}} \log e^{(\pi-\theta)\sqrt{-1}}$$

$$= \tfrac{1}{2}(\pi - \theta).$$

If $x = \dfrac{a}{u}$ where $u > 1$.

$$\therefore \frac{\pi \phi x}{C} = \tfrac{1}{2} a + \sin\left(a - \frac{a}{u}\right) + \sin\frac{a}{u} + \ldots$$

$$= \tfrac{1}{2} a + \tfrac{1}{2}\left(\pi - a + \frac{a}{u}\right) + \tfrac{1}{2}\left(\pi - \frac{a}{u}\right)$$

$$= \tfrac{1}{2} a + \pi - \tfrac{1}{2} a$$

$$= \pi\,;$$

$$\therefore\ \phi(x) = C.$$

as it ought.

But if $u < 1$,

$$\frac{\pi\phi x}{C} = \tfrac{1}{2}a - \sin\left(\frac{a}{u} - a\right) + \sin\frac{a}{u} + \ldots$$

$$= \tfrac{1}{2}a - \tfrac{1}{2}\left(\pi - \frac{a}{u} + a\right) + \tfrac{1}{2}\left(\pi - \frac{a}{u}\right)$$

$$= 0;$$

whence the series is evidently correct.

The reason of these results is obvious. It consists in the circumstance that the quantities

$$\frac{e^{\theta\sqrt{-1}} + e^{-\theta\sqrt{-1}}}{2} \text{ and } \frac{e^{\theta\sqrt{-1}} - e^{-\theta\sqrt{-1}}}{2\sqrt{-1}};$$

for $\cos\theta$ and $\sin\theta$ express the arithmetical values of these quantities only when θ is less than a right angle. In other cases they are the representation of the sine or cosine of a supplementary angle as the case may be, and we should obtain incorrect results if we were to adopt the angle itself in many deductions where these quantities are concerned.

The above verification shews this to be true by means of the two values obtained for $\frac{\pi\phi x}{C}$ according as u is greater or less than unity, these values depending on the expansion of $e^{\frac{\pi-\theta}{2}\sqrt{-1}}$ in one case, which an incorrect generalization would make $e^{\frac{\pi+\theta}{2}\sqrt{-1}}$ in the other, the correct result arising from $-e^{\frac{\pi-\theta}{2}\sqrt{-1}}$.

69. From the circumstance that these functions are of extensive utility in the application of the theory of heat, we will solve another question of the same kind.

Let us find a function (y) which represents the sides of a trapezium, viz. such that

$$\text{from } x = 0 \text{ to } x = a,\ y = cx,$$

$$\text{from } x = a \text{ to } x = \beta,\ y = ca,$$

$$x = \beta \text{ to } x = \gamma,\ y = e(\gamma - x).$$

$$\text{Assume } y = a_0 + a_1 \sin\frac{2\pi x}{\gamma} + \ldots\ldots$$

$$+ b_1 \cos\frac{2\pi x}{\gamma} + \ldots\ldots$$

$$= a_0 + a_1 \sin z + \ldots\ldots$$

$$+ b_1 \cos z + \ldots\ldots$$

$$\text{if we make } \frac{2\pi x}{\gamma} = z,$$

$$\text{suppose also, } \frac{2\pi a}{\gamma} = a',\ \frac{2\pi\beta}{\gamma} = \beta';$$

$$\text{then } fx = f\frac{\gamma}{2\pi}z = \phi(z) = y,$$

$$\text{and from } z = 0 \text{ to } z = a',\ y = \frac{c\gamma}{2\pi}.z,$$

$$z = a' \text{ to } z = \beta',\ y = ca,$$

$$z = \beta' \text{ to } z = 2\pi,\ y = \frac{c\gamma}{2\pi}(2\pi - z);$$

hence, calling $a_0\ a_1 \ldots\ldots b_1$ &c. the coefficients of $\cos 0z$ $\cos z \ldots \sin z$ &c. as in (67), we shall have for the first limit

$$\pi a_0 = \frac{1}{2}\int_0^{a'} dx\,\phi x = \frac{c\gamma}{8\pi}a'^2,$$

$$\pi a_1 = \int_0^{a'} dx \cos x\,\phi x = \frac{c\gamma}{2\pi}.(a' \sin a' - \text{vers } a'),$$

$$\pi a_2 = \int_0^{a'} dx \cos 2x\,\phi x = \frac{1}{2^2}.\frac{c\gamma}{2\pi}(2a' \sin 2a' - \text{vers } 2a'),$$

$$\pi a_3 = \int_0^{a'} dx \cos 3x\,\phi x = \frac{1}{3^2}.\frac{c\gamma}{2\pi}(3a' \sin 3a' - \text{vers } 3a'),$$

&c. = &c.

$$\pi b_1 = \int_0^{a'} dx \sin x\,\phi x = \frac{c\gamma}{2\pi}(\sin a' - a' \cos a'),$$

$$\pi b_2 = \int_0^{a'} dx \sin 2x\,\phi x = \frac{1}{2^2}.\frac{c\gamma}{2\pi}(\sin 2a' - 2a' \cos 2a'),$$

&c. = &c.

For the second limit

$$\pi a_0 = \tfrac{1}{2} c a\,(\beta' - a'),$$

$$\pi a_1 = \frac{ca}{1}(\sin \beta' - \sin a'),$$

$$\pi a_2 = \frac{ca}{2}(\sin 2\beta' - \sin 2a'),$$

&c. = &c.

$$\pi b_1 = \frac{ca}{1}(\cos a' - \cos \beta'),$$

$$\pi b_2 = \frac{ca}{2}(\cos 2a' - \cos 2\beta'),$$

&c. = &c.

For the third limit

$$\pi a_0 = \frac{1}{2} \cdot \frac{e\gamma}{2\pi} \frac{1}{2} (2\pi - \beta')^2,$$

$$\pi a_1 = -\frac{e\gamma}{2\pi} \{(2\pi - \beta') \sin \beta' + \text{vers}\beta'\},$$

$$\pi a_2 = -\frac{1}{2^2} \cdot \frac{e\gamma}{2\pi} \{2(2\pi - \beta') \sin 2\beta' + \text{vers}\, 2\beta'\},$$

&c. = &c.

$$\pi b_1 = \frac{e\gamma}{2\pi} \{(2\pi - \beta') \cos \beta' + \sin \beta'\},$$

$$\pi b_2 = \frac{1}{2^2} \cdot \frac{e\gamma}{2\pi} \{2(2\pi - \beta') \cos 2\beta' + \sin 2\beta'\},$$

&c. = &c.

Hence we obtain

$$y = \frac{c\gamma}{2\pi^2} \left\{ \begin{array}{l} \frac{a'^2}{4} \\ +(a' \sin a' - \text{vers}\, a') \frac{\cos x}{1^2} + (\sin a' - a' \cos a') \frac{\sin x}{1^2} \\ +(2a' \sin 2a' - \text{vers}\, 2a') \frac{\cos 2x}{1^2} + (\sin 2a' - 2a' \cos 2a') \frac{\sin 2x}{2^2} \\ +(3a' \sin 3a' - \text{vers}\, 3a') \frac{\cos 3x}{1^2} + (\sin 3a' - 3a' \cos 3a') \frac{\sin 3x}{3^2} \\ + \quad \&c. \qquad\qquad \&c. \end{array} \right.$$

$$+ \frac{ca}{\pi} \left\{ \begin{array}{l} \frac{1}{2}(\beta' - a') \\ +(\sin \beta' - \sin a') \frac{\cos x}{1} + (\cos a' - \cos \beta') \frac{\sin x}{1} \\ +(\sin 2\beta' - \sin 2a') \frac{\cos 2x}{2} + (\cos 2a' - \cos 2\beta') \frac{\sin 2x}{2} \\ +(\sin 3\beta' - \sin 3a') \frac{\cos 3x}{3} + (\cos 3a' - \cos 3\beta') \frac{\sin 3x}{3} \\ + \quad \&c. \qquad\qquad \&c. \end{array} \right.$$

$$+\frac{e\gamma}{2\pi^2}\begin{cases} \dfrac{(2\pi-\beta')^2}{4} \\ -\{(2\pi-\beta')\sin\beta'+\mathrm{vers}\beta'\}\dfrac{\cos x}{1^2}+\{\sin\beta'+(2\pi-\beta')\cos\beta'\}\dfrac{\sin x}{1^2} \\ -\{2(2\pi-\beta')\sin 2\beta'+\mathrm{vers}2\beta'\}\dfrac{\cos 2x}{2^2}+\{\sin 2\beta'+2(2\pi-\beta')\cos 2\beta'\}\dfrac{\sin 2x}{2^2} \\ -\{3(2\pi-\beta')\sin 3\beta'+\mathrm{vers}3\beta'\}\dfrac{\cos 3x}{3^2}+\{\sin 3\beta'+3(2\pi-\beta')\cos 3\beta'\}\dfrac{\sin 3x}{3^2} \\ -\quad \&c. \qquad\qquad \&c. \end{cases}$$

70. This is a very beautiful expression on account of its symmetry: it is however, considerably more complicated than that given by M. Fourier, and copied from his work into various others. There can be little doubt to any one who carefully examines the subject, that nearly all M. Fourier's series on this branch of the subject are erroneous. This however, must not be supposed to detract from M. Fourier, it is a mere trifle in his work.

Cor. 1. The above expression will be much simplified if $\gamma = 2\pi$, which makes $x = x$, $\alpha = \alpha'$, $\beta = \beta'$.

Cor. 2. If after making $\gamma = 2\pi$, we put $\alpha = \beta$ the figure is a triangle.

In this case the middle term of the three vanishes in the value of y, and if we further put $e = c$, or suppose the triangle isosceles we have $\alpha = \pi$: hence the equation becomes

$$y = \frac{c}{\pi}\left\{\frac{\pi^2}{2} - 4\left(\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \ldots.\right)\right\},$$

which is the equation to the isosceles triangle.

71. To verify this solution.

Let $u = \dfrac{\cos x}{1^2} + \dfrac{\cos 3x}{3^2} + \ldots\ldots$

$$\therefore \frac{du}{dx} = -\left(\frac{\sin x}{1} + \frac{\sin 3x}{3} + \ldots\ldots\right)$$

$$= -\frac{1}{4\sqrt{-1}} \log \frac{(1+e^{x\sqrt{-1}})(1-e^{-x\sqrt{-1}})}{(1+e^{-x\sqrt{-1}})(1-e^{-x\sqrt{-1}})}$$

$$= -\frac{1}{4\sqrt{-1}} \log(-1)$$

$$= -\frac{1}{4\sqrt{-1}} \log\{\cos(2n+1)\pi + \sqrt{-1}\sin(2n+1)\pi\}$$

$$= -\frac{1}{4\sqrt{-1}} \log e^{(2n+1)\pi\sqrt{-1}}$$

$$= -\frac{(2n+1)\pi}{4} \text{ if } x < \pi;$$

$$\therefore u = -\frac{(2n+1)\pi}{4} x + C,$$

$$0 = -(2n+1)\frac{\pi^2}{8} + C;$$

$$\therefore C = (2n+1)\frac{\pi^2}{8};$$

$$\therefore u = (2n+1)\left(\frac{\pi^2}{8} - \frac{\pi x}{4}\right);$$

but when $x = 0$.

$$u = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots. = \frac{\pi^2}{8};$$

$$\therefore n = 0,$$

and $u = \frac{\pi^2}{8} - \frac{\pi x}{4}$ is the sum;

$$\text{but } y = \frac{c}{\pi}\left(\frac{\pi^2}{2} - 4u\right)$$

$$= \frac{c}{\pi}\left(\frac{\pi^2}{2} - \frac{\pi^2}{2} + \pi x\right)$$

$$= cx.$$

But if $x > \pi$,

$\frac{dx}{du}$ changes its sign, and we get

$$u = (2n+1)\frac{\pi}{4}x + C,$$

$$0 = (2n+1)\frac{3\pi^2}{8} + C,$$

$$u = (2n+1)\frac{\pi}{4}\left(x - \frac{3\pi}{2}\right),$$

but when $x = 2\pi$, $u = \frac{\pi^2}{8}$; $\therefore n = 0$,

$$\text{and } y = \frac{c}{\pi}(2\pi^2 - \pi x)$$

$$= c(2\pi - x),$$

which is the complete verification of the formula.

Motion of heat in the ring.

72. The equation of motion is

$$\frac{dv}{dt} = \frac{K}{CD}\frac{d^2v}{dx^2} - \frac{hl}{CDS}v$$

$$= k\frac{d^2v}{dx^2} - \lambda v,$$

if $k = \frac{K}{CD}$, $\lambda = \frac{hl}{CDS}$.

Put $v = e^{-\lambda t} . u$;

$$\therefore \frac{dv}{dt} = e^{-\lambda t}\left(\frac{du}{dt} - \lambda u\right),$$

$$\frac{d^2 v}{dx^2} = e^{-\lambda t}\frac{d^2 u}{dx^2};$$

$$\therefore e^{-\lambda t}\left(\frac{du}{dt} - \lambda u\right) = e^{-\lambda t}\left(k\frac{d^2 u}{dx^2} - \lambda u\right),$$

$$\frac{du}{dt} = k\frac{d^2 u}{dx^2}.$$

{For solution see Gray & Walton p 354}

Now it is clear that when x is increased by $2\pi r$; r being the radius of the ring, u must remain unaltered,

$$u = f\left\{\sin\frac{nx}{r},\ \cos\frac{nx}{r},\ t\right\}.$$

Let one term be $u = a\sin\frac{nx}{r}\phi(t)$,

$$\text{then } \phi' t = -\frac{kn^2}{r^2}\phi t,$$

$$\frac{d\log\phi t}{dt} = -\frac{n^2}{r^2}k,$$

$$\phi(t) = e^{-\frac{kn^2}{r^2}t};$$

$$\therefore u = ae^{-\frac{kn^2}{r^2}t}\sin\frac{nx}{r}.$$

The same results by taking $\cos\frac{nx}{r}$ instead of the corresponding sine,

$$\text{hence } u = \Sigma e^{-\frac{kn^2}{r^2}t}\left(b\cos\frac{nx}{r} + a\sin\frac{nx}{r}\right),$$

after a very considerable time it is evident that the terms which make $n > 1$, are smaller than those which make $n =$ or < 1, in this case

$$u = c + e^{-\frac{kt}{r^2}}\left(a \sin\frac{x}{r} + b\cos\frac{x}{r}\right);$$

$$\therefore\ v = \left\{c + e^{-\frac{kt}{r^2}}\left(a\sin\frac{x}{r} + b\cos\frac{x}{r}\right)\right\}e^{-\lambda t}.$$

Let v' be the value of v at a point distant from this by a semi-circle, so that $x + \pi r$ must be put for x;

$$\therefore\ v' = \left\{c + e^{-\frac{kt}{r^2}}\left(-a\sin\frac{x}{r} - b\cos\frac{x}{r}\right)\right\}e^{-\lambda t};$$

$$\therefore\ \frac{v + v'}{2} = c\,.\,e^{-\lambda t}.$$

Hence it appears that the sum of the temperatures at the extremities of any diameter is the same for every diameter.

72. PROB. To find the mean temperature.

Since $\begin{matrix}\sin\\ \cos\end{matrix}\frac{nx}{r}$ go through all their values, whilst x goes through $2\pi r$, it is evident that the terms involving circular forms will not appear in the mean temperature.

Hence, mean temperature $= ce^{-\lambda t}$.

COR. This is the same expression as we found for the semi-sum of the two temperatures at the extremity of any diameter.

We have in this result a form which can very readily be put to the test of experiment, and it has been done by M. Fourier.

A ring was unequally heated and suffered to cool; four thermometers were placed in it, a pair in the two extremities of one diameter, the other pair in those of another.

The temperatures were observed repeatedly. The semi-sum of the temperatures from one pair sometimes exceeded, and at others fell short of that of the other, but by quantities so small, that they were abundantly accounted for by the necessary errors of operation: the following set of results is a specimen,

temperature of one pair of thermometers $43\frac{5}{6}$, $34\frac{3}{8}$,

.....another do 41, 38.

Semi-sum of 1st = $39\frac{5}{48}$,

..... 2d = $39\frac{1}{2}$;

difference = $\frac{19}{48}$,

which is less than $\frac{1}{2}$.

A large number of results equally close were obtained and no discrepancies.

73. Hitherto we have applied our calculations only on the hypothesis, that the radiating power or exterior conductivity varies as the excess of temperature of the body above that of the external atmosphere. This law, it is obvious, is only an approximate one, hence it becomes a question, how do we obtain such an accurate verification of our results? We shall shew presently, that the verification has been applied to them in a form which they do not owe to the law of radiation, and that Dulong and Petit's law leads to precisely the same conclusions. The accurate solution of the problem adopting that law has never been attempted except in one case, that of M. Libri's memoir; in that case the author appears to have forgotten that the conclusions deduced from a particular solution, consisting of a single term of the series, only

applies to linear equations: his results are consequently incorrect. We have attempted to give a solution which will at least satisfy the above query, which by the way is all that we require of it.

The equation is $$\frac{dv}{dt} = K\frac{d^2v}{dx^2} - \lambda v,$$

where Newton's law is adopted, CD being put equal to unity.

The substitution of Dulong's and Petit's law makes

$$\frac{dv}{dt} = K\frac{d^2v}{dx^2} - \mu(a^v - 1).$$

Now $a^v = e^{av}$ if $a = \log_e a$,

which is a small quantity since $a = 1.165$, v being expressed in degrees of the centigrade thermometer.

Hence, if we expand v we obtain

$$\frac{dv}{dt} = K\frac{d^2v}{dx^2} - \mu\left(av + \frac{a^2v^2}{1.2} + \frac{a^3v^3}{1.2.3} + \ldots\ldots\right).$$

Suppose now $v = V + aV_1 + a^2V_2 + \ldots\ldots$, by putting $\mu a = \lambda$, we obtain

$$\frac{dV}{dt} + a\frac{dV_1}{dt} + \ldots = K\left(\frac{d^2V}{dx^2} + a\frac{d^2V_1}{dx^2} + \ldots\ldots\right) -$$

$$\lambda\left\{V + aV_1 + \ldots + \frac{a}{1.2}(V + aV_1 + \ldots)^2 + \ldots\ldots\right\}.$$

But since the hypothesis can be made good by interchanging and varying the values of V, V_1, ... in any manner, we may so take these quantities that

$$\frac{dV}{dt} = K\frac{d^2V}{dx^2} - \lambda V,$$

$$\frac{dV_1}{dt} = K\frac{d^2V_1}{dx^2} - \lambda\left(V_1 + \frac{V^2}{2}\right),$$

&c. = &c.

But it will be a most complex operation to attempt the simultaneous solution of these equations, since they are no longer linear, and consequently we cannot discuss one term separately from the rest.

74. To obtain the state of temperature after a long period, let us retain only the three first terms, and assume

$$V = ce^{-\lambda t} + \left(a\sin\frac{x}{r} + b\cos\frac{x}{r}\right)e^{-\left(\frac{K}{r^2}+\lambda\right)t}.$$

As a further simplification, requisite indeed to the assumption that these two terms give the integral with sufficient accuracy, assume that $\frac{K}{r^2}$ is much greater than λ.

$$\therefore\ V^2 = c^2e^{-2\lambda t} + 2ce^{-\left(\frac{K}{r^2}+2\lambda\right)t}\left(a\sin\frac{x}{r} + \cos\frac{x}{r}\right);$$

$$+ e^{-\left(\frac{2K}{r^2}+2\lambda\right)t}\left(\frac{a^2+b^2}{2} - \frac{a^2-b^2}{2}\cos\frac{2x}{r} + \ldots\ldots\right)$$

$$= c^2e^{-2\lambda t} + 2ce^{-\left(\frac{K}{r^2}+2\lambda\right)t}\left(a\sin\frac{x}{r} + b\cos\frac{x}{r}\right) \text{ nearly};$$

$$\therefore\ \frac{dV_1}{dt} = K\frac{d^2V_1}{dx^2} - \lambda V_1 - \frac{\lambda}{2}c^2e^{-2\lambda t};$$

$$- \lambda ce^{-\left(\frac{K}{r^2}+2\lambda\right)t}\left(a\sin\frac{x}{r} + b\cos\frac{x}{r}\right).$$

If $V_1 = \left(a_1\sin\frac{x}{r} + b_1\cos\frac{x}{r}\right)e^{-\left(\frac{K}{r^2}+\lambda\right)t} + c_1e^{-\lambda t} + P;$

it is obvious that the term P only is to be retained since the quantities aa_1 &c. may be added to a..... in the expression $v = V + aV_1 + ...$ and will thus only alter the values of the arbitrary constants.

$$\text{Let } \therefore V_1 = e^{-2\lambda t} . P + e^{-\left(\frac{K}{r^2} + 2\lambda\right)t} . Q;$$

$$\therefore \frac{dV_1}{dt} = -2\lambda e^{-2\lambda t} P - \left(\frac{K}{r^2} + 2\lambda\right) e^{-\left(\frac{K}{r^2} + 2\lambda\right)t} . Q.$$

$$+ e^{-2\lambda t} . \frac{dP}{dt} + e^{-\left(\frac{K}{r^2} + 2\lambda\right)t} . \frac{dQ}{dt};$$

$$\therefore -2\lambda P = K\frac{d^2P}{dx^2} - \lambda P - \frac{\lambda}{2}c^2.$$

$$-\left(\frac{K}{r^2} + 2\lambda\right) Q = K\frac{d^2Q}{dx^2} - \lambda Q - \lambda c\left(a \sin\frac{x}{r} + b\cos\frac{x}{r}\right);$$

$$\therefore \frac{d^2P}{dx^2} + \frac{\lambda}{K}P - \frac{\lambda c^2}{2K} = 0;$$

$$P = \frac{c^2}{2} + A \sin\sqrt{\frac{\lambda}{K}} . x + B\cos\sqrt{\frac{\lambda}{K}} . x;$$

$$\frac{d^2Q}{dx^2} + \left(\frac{K}{r^2} + \lambda\right)\frac{Q}{K} - \frac{\lambda c}{K}\left(a \sin\frac{x}{r} + b\cos\frac{x}{r}\right) = 0;$$

$$Q = C_1 \sin\frac{x}{r} + D_1\cos\frac{x}{r} + C\sin\sqrt{\frac{\frac{K}{r^2} + \lambda}{K}} . x + D\cos\sqrt{\frac{\frac{K}{r^2} + \lambda}{K}} . x;$$

$$\text{where } -\frac{C_1}{r^2} + \left(\frac{K}{r^2} + \lambda\right)\frac{C_1}{K} - \frac{\lambda ac}{K} = 0;$$

$$-\frac{D_1}{r^2} + \left(\frac{K}{r^2} + \lambda\right)\frac{D_1}{K} - \frac{\lambda bc}{K} = 0;$$

$$C_1 = \frac{\lambda ac}{\frac{K}{r^2} + \lambda - \frac{K}{r^2}} = ac; \quad D_1 = \frac{\lambda bc}{\frac{K}{r^2} + \lambda - \frac{K}{r^2}} = bc;$$

$$\therefore\ Q = c\left(a \sin \frac{x}{r} + b \cos \frac{x}{r}\right) + C \sin \sqrt{\frac{1}{r^2} + \frac{\lambda}{K}}\, x$$

$$+ D \cos \sqrt{\frac{1}{r^2} + \frac{\lambda}{K}}\, x;$$

$$\therefore\ V_1 = e^{-2\lambda t}\left(\frac{c^2}{2} + A \sin \sqrt{\frac{\lambda}{K}}\, x + B \cos \sqrt{\frac{\lambda}{K}}\, x\right)$$

$$+ e^{-\left(\frac{K}{r^2} + 2\lambda\right)t}\left\{c\left(a \sin \frac{x}{r} + b \cos \frac{x}{r}\right) + C \sin \sqrt{\frac{1}{r^2} + \frac{\lambda}{K}}\, . x\right.$$

$$\left. D \cos \sqrt{\frac{1}{r^2} + \frac{\lambda}{K}}\, x\right\};$$

$$\text{hence } v = ce^{-\lambda t} + \left(a \sin \frac{x}{r} + b \cos \frac{x}{r}\right) e^{-\left(\frac{K}{r^2} + \lambda\right)t}$$

$$+ ae^{-2\lambda t}\left(\frac{c^2}{2} + A \sin \sqrt{\frac{\lambda}{K}}\, x + B \cos \sqrt{\frac{\lambda}{K}}\, x\right)$$

$$+ ae^{-\left(\frac{K}{r^2} + 2\lambda\right)t}\left\{c\left(a \sin \frac{x}{r} + b \cos \frac{x}{r}\right) + C \sin \sqrt{\frac{1}{r^2} + \frac{\lambda}{K}}\, x\right.$$

$$\left. + D \cos \sqrt{\frac{1}{r^2} + \frac{\lambda}{K}}\, x\right\}.$$

Now from the nature of the case v will remain unchanged by the substitution of $2\pi r + x$ for x whatever be t and x; this shews that $A = 0$, $B = 0$, $C = 0$, $D = 0$;

$$\therefore\ v = ce^{-\lambda t} + \left(a \sin \frac{x}{r} + b \cos \frac{x}{r}\right) e^{-\left(\frac{K}{r^2} + \lambda\right)t} . (1 + ace^{-\lambda t})$$

$$+ \frac{ac^2}{2} e^{-2\lambda t} . (A).$$

75. We are now able to obtain the *general* solution. It is obvious from the preceding process that it will consist of an infinite series of terms in which not only $\frac{K}{r^2}$ will be affected with a coefficient, but λ will be so too. In fact, if we assume the solution to be

$$v = c_1 e^{-\lambda t} + c_2 e^{-2\lambda t} + \ldots\ldots$$

$$+ \left(a_1 \sin \frac{x}{r} + b_1 \cos \frac{x}{r}\right) \left\{A_1 e^{-\left(\frac{K}{r^2}+\lambda\right)t} + B_1 e^{-\left(\frac{nK}{r^2}+n\lambda\right)t} + \ldots\right\}$$

$$+ \left(a_2 \sin \frac{2x}{r} + b_1 \cos \frac{2x}{r}\right) \left\{A_2 e^{-\left(\frac{mK}{r^2}+n\lambda\right)t} + \ldots\right\}$$

and substitute this value of v in the given equation, we know that the result can be ranged under the form

$$0 = \Sigma_{p,\,l} \left(a_p \sin \frac{px}{r} + b_p \cos \frac{px}{r}\right) e^{-l.t};$$

and we shall be at liberty to assume the coefficient of every power of e separately equal 0, thus obtaining as many equations for the determination of the coefficient as there are combinations of the quantities

$$\lambda,\ 2\lambda \ldots \text{ with } \frac{K}{r^2},\ \frac{2K}{r^2} + \ldots$$

Also quantities c_1, $c_2 \ldots$ will be determinate from the circumstance that none of the quantities a_1, A_1 &c. will enter as coefficients of the term $e^{-n\lambda t}$, and there are as many values of n as of c, whence the number of equations will equal that of the unknown quantities. We are thus in possession of a complete solution.

76. Cor. If v_1 v_2 represent the temperature at the points x, $\pi r + x$, after a long interval

$$\frac{v_1 + v_2}{2} = ce^{-\lambda t} + \frac{ac^2}{2} e^{-2\lambda t} = u; \text{ suppose}$$

$$\therefore \frac{du}{dt} = -\lambda (ce^{-\lambda t} + ac^2 e^{-2\lambda t})$$

$$= -\lambda ce^{-\lambda t} . (1 + ace^{-\lambda t});$$

hence the effect of taking Dulong's law is that we find the velocity of cooling diminishes more rapidly than would be obtained by Newton's law. Nothing else is altered.

77. PROB. To find the temperature at any time, from knowing the initial temperature.

Let $f(x)$ represent the initial temperature;

then putting $t = 0$,

$$f(x) = c + \left(a_1 \sin \frac{x}{r} + b_1 \cos \frac{x}{r}\right)$$

$$+ \left(a_2 \sin \frac{2x}{r} + b_2 \cos \frac{2x}{r}\right) + \ldots$$

$$\therefore \int_0^{2\pi r} f(x)\, dx = 2\pi cr,$$

$$\int_0^{2\pi r} f(x) \sin \frac{nx}{r}\, dx = a_n \pi r;$$

$$\therefore \pi r f(x) = \tfrac{1}{2} \int_0^{2\pi r} fx\, dx + \sin \frac{x}{r} \int_0^{2\pi r} fx . \sin \frac{x}{r} dx + \ldots$$

and $\pi r v = \frac{1}{2} e^{-\lambda t} \int_0^{2\pi r} fx . dx + e^{-\left(\frac{k}{r^2} + \lambda\right) t} \sin \frac{x}{r} \int_0^{2\pi r} fx \sin \frac{x}{r} dx$

$$+ e^{-\left(\frac{2^2 k}{r^2} + \lambda\right) t} \sin \frac{2x}{r} \int_0^{2\pi r} fx \sin \frac{2x}{r} dx + \ldots\ldots$$

$$+ e^{-\left(\frac{k}{r^2} + \lambda\right) t} \cos \frac{x}{r} \int_0^{2\pi r} fx \cos \frac{x}{r} dx + \ldots\ldots$$

which gives us the complete value of the temperature at any time.

78. Prob. To find the temperature of a ring which has been heated by a focus in one point, and is then left to cool.

The general equation is

$$\pi r v = e^{-\lambda t}\left\{\tfrac{1}{2}\int_0^{2\pi r} f x\,dx + e^{-\frac{kt}{r^2}}\left(\sin\frac{x}{r}\int_0^{2\pi r}\sin\frac{x}{r} f x\,dx\right.\right.$$

$$\left. + \cos\frac{x}{r}\int_0^{2\pi r}\cos\frac{x}{r} f x\,dx\right)$$

$$+ e^{-\frac{2^2 kt}{r^2}}\left(\sin\frac{2x}{r}\int_0^{2\pi r}\sin\frac{2\pi x}{r} f x\,dx + \cos\frac{2x}{r}\int_0^{2\pi r}\cos\frac{2\pi x}{r} f x\,dx\right)$$

$$\left. + \&c. \ldots\ldots\ldots\ldots\ldots\ldots\right\}$$

where $f(x)$ represents the initial state.

Now the initial state is determined by the equation

$$v = ae^{ax} + be^{-ax} \quad (59);$$

but from the nature of the case, we must have $\frac{dv}{dx}=0$, when $x = \pi r$, the focus being in the origin;

$$\therefore\ ae^{a\pi r} = be^{-a\pi r};$$

$$v = be^{-ax} + be^{-2a\pi r + ax}$$

$$= be^{-a\pi r}\left\{e^{(-\pi r + x)a} + e^{(\pi r - x)a}\right\}$$

$$= be^{-a\pi r}\left\{e^{(\pi r - x)a} + e^{-(\pi r - x)a}\right\} = fx;$$

$$\therefore\ \int_0^{2\pi r} f x\,dx = \frac{2b}{a}e^{-a\pi r}\left(e^{a\pi r} - e^{-a\pi r}\right);$$

$$\int \sin n\frac{x}{r}\, e^{ax}\, dx = -\frac{n}{ar}\int e^{ax}\cos n\frac{x}{r}\, dx + \&c.$$

$$= -\frac{n}{a^2 r}\, e^{ax}\cos\frac{nx}{r} - \frac{n^2}{a^2 r^2}\int e^{ax}\sin\frac{nx}{r}\, dx$$

$$= -\frac{nr}{a^2 r^2 + n^2}\, e^{ax}\cos\frac{nx}{r}$$

$$= -\frac{nr}{a^2 r^2 + n^2}\left(e^{2\pi a r} - 1\right);$$

$$\int \sin\frac{nx}{r}\, e^{-ax}\, dx = +\frac{nr}{a^2 r^2 + n^2}\left(1 - e^{-2\pi a x}\right);$$

$$\therefore \int_0^{2\pi r} \sin\frac{nx}{r}\left(e^{\pi r a - ax} + e^{-\pi r a + ax}\right) dx = \frac{nr}{a^2 r^2 + n^2}\begin{cases} e^{\pi r a} - e^{-\pi r a} \\ -e^{\pi r a} + e^{-\pi r a} \end{cases}$$

$$= 0;$$

$$\int \cos\frac{nx}{r}\, e^{ax}\, dx = \frac{e^{ax}}{a}\cos\frac{nx}{r} + \frac{n}{ra}\int e^{ax}\sin\frac{nx}{r}\, dx$$

$$= \frac{1}{a}\left(e^{2\pi r a} - 1\right) - \frac{n^2}{a\left(a^2 r^2 + n^2\right)}\left(e^{2\pi a r} - 1\right)$$

$$\int \cos\frac{nx}{r}\, e^{-ax}\, dx = +\frac{1}{a}\left(1 - e^{-2\pi r a}\right) - \frac{n^2}{a\left(a^2 r^2 + n^2\right)}\left(1 - e^{-2\pi a r}\right);$$

$$\therefore \int_0^{2\pi}\left(e^{\pi r a - ax} + e^{-\pi r a + ax}\right).\cos\frac{nx}{r}\, dx = \left(e^{\pi r a} - e^{-\pi r a}\right)$$

$$\left\{\frac{2}{a} - \frac{2n^2}{a\left(a^2 r^2 + n^2\right)} = \frac{2a^2 r^2}{a\left(a^2 r^2 + n^2\right)} = \frac{2ar^2}{a^2 r^2 + n^2}\right\};$$

hence we obtain

$$\pi r v = b e^{-a\pi r}\left(e^{\pi r a} - e^{-\pi r a}\right) e^{-\lambda t}\left(\frac{1}{a} + \frac{2ar^2}{a^2 r^2 + 1}\, e^{-\frac{kt}{r^2}}\cos\frac{x}{r}\right.$$

$$\left. + \frac{2ar^2}{a^2 r^2 + 2^2}\, e^{-\frac{2^2 kt}{r^2}}\cos\frac{2x}{r} + \ldots\right).$$

Now if M be the mean initial temperature,

$$M = \int_0^{2\pi r} \frac{be^{-a\pi r}}{2\pi r} \left\{ e^{(\pi r - x)a} + e^{-(\pi r - x)a} \right\} dx$$

$$= \frac{be^{-a\pi r}}{2\pi a r} \left(e^{-\pi r a + ax} - e^{\pi r a - ax} \right)$$

$$= \frac{be^{-a\pi r}}{\pi a r} \left(e^{+\pi r a} - e^{-\pi r a} \right);$$

$$\therefore\ v = 2Me^{-\lambda t} \left(\frac{1}{2} + \frac{a^2 r^2}{a^2 r^2 + 1} e^{-\frac{kt}{r^2}} \cos \frac{x}{r} \right.$$

$$\left. + \frac{a^2 r^2}{a^2 r^2 + 2^2} e^{-\frac{2^2 kt}{r^2}} \cos \frac{2x}{r} + \frac{a^2 r^2}{a^2 r^2 + 3^2} e^{-\frac{3^2 kt}{r^2}} \cos \frac{3x}{r} + \ldots \right),$$

the temperature required.

79. Prob. To find the motion of heat in a sphere.

The equation of motion is

$$\frac{dv}{dt} = k \left(\frac{d^2 v}{dx^2} + \frac{2}{x} \frac{dv}{dx} \right),$$

subject to the condition $\frac{dv}{dx} + h'v = 0$; $h' = \frac{h}{K}$, when $x = X$, where x is the radius of any section, X the radius of the sphere,

$$k = \frac{K}{CD};$$

let $y = vx$; $\quad \therefore\ v = \frac{y}{x}$;

$$\therefore \frac{dv}{dx} = \frac{dy}{x dx} - \frac{y}{x^2},$$

$$\frac{d^2 v}{dx^2} = \frac{d^2 y}{x dx^2} - \frac{2dy}{x^2 dx} + \frac{2y}{x^3},$$

$$\frac{dv}{dt} = \frac{1}{x} \frac{dy}{dt};$$

$$\therefore \frac{dy}{dt} = k\left(\frac{d^2y}{dx^2} - \frac{2dy}{xdx} + \frac{2y}{x^2} + \frac{2dy}{xdx} - \frac{2y}{x^2}\right)$$

$$= k\frac{d^2y}{dx^2},$$

the solution of which is (7~~2~~) 2/

$$y = e^{-kn^2t}(a\cos nx + b\sin nx);$$

$$\therefore v = \frac{e^{-kn^2t}}{x}(a\cos nx + b\sin nx).$$

Now the temperature at the center is supposed finite: hence when $x = 0$, v is finite; $\therefore$ a must $= 0$;

$$\therefore v = be^{-kn^2t}\frac{\sin nx}{x},$$

$= 0 \therefore$

$$\text{and } \frac{dv}{dx} + h'v\frac{nx\cos nx - \sin xn}{x^2} + h'\frac{\sin nx}{x} = 0,$$

when $x = X$;

$$nX\cos nX - \sin nX + h'X\sin nX = 0,$$

$$nX = (1 - h'X)\tan nX,$$

$$\frac{\tan nX}{nX} = \frac{1}{1 - h'X}, \quad \text{or if } nX = \epsilon,$$

$$\frac{\tan\epsilon}{\epsilon} = c, \quad \text{a constant quantity.}$$

This will give an infinite number of values of ϵ, and consequently of n.

Let n_1, n_2,... be the values of n;

$$\therefore vx = a_1e^{-kn_1^2t}\sin n_1x + \ldots\ldots\ldots$$

if $t = 0$,

$$v_0 = \frac{a_1}{x}\sin n_1x + \ldots\ldots\ldots$$

is the initial temperature.

80. To determine a_1,.....

Let $f(x)$ represent the initial temperature,

then $xF(x) = a_1 \sin n_1 x + ...$

$$\int_0^X \sin mx \sin n_1 x\, dx = -\frac{1}{m}\cos mx \sin n_1 x + \frac{n_1}{m}\int \cos mx \cos n_1 x\, dx$$

$$= -\frac{1}{m}\cos mx \sin n_1 x + \frac{n_1}{m^2}\sin mx \cos n_1 x + \frac{n_1^2}{m^2}\int \sin mx \sin n_1 x\, dx$$

$$= \frac{-m \cos mx \sin n_1 x + n_1 \sin mx \cos n_1 x}{m^2 - n_1^2}$$

$$= \frac{1}{m^2 - n_1^2}(-m \cos mX \sin n_1 X + n_1 \sin mX \cos n_1 X).$$

Now n_1 and m being each determined by the equation

$$\frac{n_1 X}{\tan n_1 X} = 1 - h'X,$$

$$\frac{mX}{\tan mX} = 1 - h'X,$$

$$\frac{n_1}{m} = \frac{\tan n_1 X}{\tan mX},$$

$$n_1 \sin mX \cos n_1 X = m \sin n_1 X \cos mX;$$

hence $\int_0^X \sin mx \cos nx\, dx = 0$, except $m = n_1$.

In the last case,

$$\int_0^X \sin^2 n_1 x\, dx = \tfrac{1}{2}\int_0^X (1 - \cos 2n_1 X)$$

$$= \tfrac{1}{2}X - \frac{1}{4n_1}\sin 2n_1 X;$$

$$\therefore \int_0^X xFx \sin n_1 x\, dx = a_1\left(\tfrac{1}{2}X - \frac{1}{4n_1}\sin 2n_1 X\right),$$

$$a_1 = \frac{2\int_0^X x F(x) \sin n_1 x \, dx}{X - \frac{1}{2n_1} \sin 2n_1 X};$$

$$\therefore \frac{vx}{2} = \Sigma e^{-kn_1^2 t} \sin n_1 x . \frac{\int_0^X x F x \sin n_1 x \, dx}{X - \frac{1}{2n_1} \sin 2n_1 X} *.$$

81. Suppose the sphere to have at first an uniform temperature = 1; $\therefore F(x) = 1$,

$$\int_0^X x F x \sin n_1 x \, dx = \int_0^X x \sin n_1 x \, dx$$

$$= -\frac{x}{n_1} \cos n_1 x + \frac{1}{n_1^2} \sin n_1 x + \ldots.$$

$$= \frac{1}{n_1^2} (\sin n_1 X - X n_1 \cos n_1 X)$$

$$= \frac{X^2}{\epsilon_1^2} (\sin \epsilon_1 - \epsilon_1 \cos \epsilon_1);$$

$$\therefore a_1 = 2 \frac{\frac{X^2}{\epsilon_1^2} (\sin \epsilon_1 - \epsilon_1 \cos \epsilon_1)}{X - \frac{X}{2\epsilon_1} \sin 2\epsilon_1}$$

$$= \frac{2}{\epsilon_1} X \left(\frac{\sin \epsilon_1 - \epsilon_1 \cos \epsilon_1}{\epsilon_1 - \sin \epsilon_1 \cos \epsilon_1} \right)$$

$$= \frac{2}{\epsilon_1} X \left(\frac{h' X \sin \epsilon_1}{\epsilon_1 - \sin \epsilon_1 \cos \epsilon_1} \right)$$

$$= \frac{2}{\epsilon_1} . \frac{h' X^2}{\epsilon_1 \operatorname{cosec} \epsilon_1 - \cos \epsilon_1};$$

* It will be observed that I have contented myself with Fourier's solution. M. Poisson, on arriving at the same conclusion, observes that Fourier has not proved, directly or indirectly, the possibility of representing $f(r)$ by the series which results from making $t = 0$. The objection of course rests on the form being that of a number of circular functions; but M. Poisson does not appear, as far as I can see, to get over the difficulty. See *Theorie de la Chaleur*, c. VIII. et IX.

$$\therefore\ v x = \Sigma . e^{-k n_1^2 t} \sin n_1 x . \frac{2}{\epsilon_1} \frac{h' X^2}{\epsilon_1 \operatorname{cosec} \epsilon_1 - \cos \epsilon_1} .$$

Restore to k its value $\frac{K}{CD}$; to n_1 its value $\frac{\epsilon_1}{X}$; and to h' its value $\frac{h}{K}$ and we get

$$v = \frac{2hX}{K} \Sigma_{\epsilon_1}^{\epsilon_\infty} . \left\{ e^{-\frac{K}{CD} \cdot \frac{\epsilon_1^2}{X^2} \cdot t} \frac{\sin \frac{\epsilon_1 x}{X}}{\frac{\epsilon_1 x}{X}} . \frac{1}{\epsilon_1 \operatorname{cosec} \epsilon_1 - \cos \epsilon_1} \right\},$$

which is the most simple form to which the expression can be reduced.

82. If h or the radius X is very small the least value of ϵ deduced from the equation $\frac{\epsilon}{\tan \epsilon} = 1 - \frac{hX}{K}$ is very near zero, and thus

$$\frac{\epsilon (1 - \frac{1}{2} \epsilon^2)}{\epsilon (1 - \frac{1}{6} \epsilon^2)} = 1 - \tfrac{1}{3} \epsilon^2 ;$$

$$\therefore\ \epsilon^2 = 3 \frac{h}{K} X.$$

$$\text{Also } \epsilon \operatorname{cosec} \epsilon - \cos \epsilon = (1 + \frac{1}{6} \epsilon^2) - (1 - \frac{1}{2} \epsilon^2)$$

$$= \frac{2}{3} \epsilon^2 = 2 \frac{hX}{K},$$

$$\text{and } \frac{\sin \frac{\epsilon x}{X}}{\frac{\epsilon x}{X}} = 1,$$

hence by substitution

$$v = \frac{2hX}{K} \cdot \frac{e^{-\frac{3h}{CDX} \cdot t}}{\frac{2hX}{K}} + \ldots\ldots$$

$$= e^{-\frac{3ht}{CDX}} \text{ nearly;}$$

the other terms being very small compared with the first.

Hence, in this case the temperature at every point is the same at a given time.

Cor. 1. The times which different spheres of the same substance occupy in losing an nth part of their temperature are as their radii.

Cor. 2. If v_1, v_2 correspond to t_1, t_2 in any body, we have

$$\log v_2 - \log v_1 = -\frac{3ht_2}{CDX} + \frac{3h}{CDX} \cdot t_1,$$

$$\text{hence } \frac{3h}{CDX} = \frac{\log v_1 - \log v_2}{t_2 - t_1}.$$

83. The result of the last proposition applies to any small body, whatever its shape, provided all its points preserve the same temperature,

if S be the exterior surface,

V be the volume of the body,

CDV is the quantity of heat requisite to raise the temperature from 0 to 1.

$hSvdt$ is the quantity which escapes by the surface in dt;

$$\therefore -CDVdv = +hSvdt,$$

$$\text{or } \frac{dv}{dt} = -\frac{hSv}{CDV},$$

$$v = e^{-\frac{hS}{CDV}\cdot t}.$$

Cor. The second corollary of the last proposition combined with the present proposition, enables us to find the radiating power in terms of the specific heat, so that if the one be known the other can be determined. The result was known from experiment before it was given by theory.

84. To find the error of a thermometer.

Suppose it plunged into a warm liquid.

Let v = the temperature indicated by the thermometer, i. e. the number of degrees by which it is more elevated than in the surrounding air, u the temperature of the liquid, h, H the coefficients of cooling of the thermometer and liquid,

$$\text{then } dv = -h(v-u)\,dt,$$

$$du = -Hu\,dt;$$

$$\therefore\ u = ae^{-Ht},$$

$$\frac{dv}{dt} = -hv + ahe^{-Ht},$$

$$ve^{ht} = h\int dt e^{(h-H)t}$$

$$= \frac{ha}{h-H}e^{(h-H)t} + b.$$

The initial value of u is evidently a.

Let that of $v-u$ be Δ,

$$\text{then } v-u = \frac{ha}{h-H}e^{-Ht} + be^{-ht} - ae^{-Ht} = \frac{Ha}{h-H}e^{-Ht} + be^{-ht},$$

$$\Delta = \frac{Ha}{h-H} + b;$$

$$\therefore\ v - u = \frac{Ha}{h-H}e^{-Ht} + \left(\Delta - \frac{Ha}{h-H}\right)e^{-ht}$$

$$= \Delta e^{-ht} + \frac{Ha}{h-H}(e^{-Ht} - e^{-ht}),$$

the quantity $v - u$ is the error of the thermometer.

Cor. After a considerable time has elapsed the error is

$$\frac{Ha}{h-H}e^{-Ht} = \frac{Hu}{h-H}.$$

85. It may be remarked that the verification of the laws to which we have arrived for the cooling of a sphere has been performed with great accuracy, and the results are perfectly satisfactory; to extract them in the present work is hardly requisite.

86. Prob. To find the mean temperature of a sphere, that is the temperature which it would have if the whole heat were uniformly distributed over it.

Let v = temperature at distance x from the center,

$4\pi\int x^2 dx \,.\, v$ is the whole heat,

and $\frac{4\pi X^3}{3}$ the whole volume;

$$\therefore \text{ mean temperature} = \frac{3}{X^3}\int_0^X x^2 v\, dx.$$

Now $\int x^2 \frac{\sin nx}{nx}\, dx = \frac{1}{n}\int x \sin nx\, dx$

$$= -\frac{1}{n^2}x\cos nx + \frac{1}{n^2}\int \cos nx\, dx$$

$$= -\frac{1}{n^2}x\cos nX + \frac{1}{n^3}\sin nX$$

$$= \frac{X^3}{\epsilon^3}(\sin\epsilon - \epsilon\cos\epsilon),$$

hence the value V of the mean temperature is,

$$V = 3\frac{2hX}{KX^3}\Sigma\left\{e^{-\frac{K}{CD}\frac{\epsilon^2}{X^2}\cdot t}\frac{X^3}{\epsilon^3}\cdot\frac{\sin\epsilon-\epsilon\cos\epsilon}{\epsilon\,\mathrm{cosec}\,\epsilon-\cos\epsilon}\right\}$$

$$= 12\frac{h}{K}\Sigma\left\{e^{-\frac{K}{CD}\frac{\epsilon^2}{X^2}t}\frac{(\sin\epsilon-\epsilon\cos\epsilon)\,X\sin\epsilon}{\epsilon^3(2\epsilon-\sin 2\epsilon)}\right\}$$

$$= 12\Sigma\left\{e^{-\frac{K}{CD}\frac{\epsilon^2}{X^2}t}\cdot\frac{(\sin\epsilon-\epsilon\cos\epsilon)^2}{\epsilon^3(2\epsilon-\sin 2\epsilon)}\right\}.$$

CHAPTER IV.

ON GAS AND VAPOUR.

87. THE laws which regulate the expansion and other changes of gaseous bodies, are in many respects so remarkable that we have devoted an entire Chapter to them. The formulæ which have been obtained for the velocity of cooling of bodies, are accurate only when the temperatures are measured by the air thermometer, except when the variations of temperature are small. The remarkable simplicity of the law induces the persuasion that air expands by equal volumes for equal increments of temperature.

There is indeed no absolute certainty that in experiments made with the thermometers of air and mercury, the error does not lie in the air thermometer; but the presumptive evidence is so strong against it, that we look on the fact as beyond dispute. For, the nature of gaseous bodies is such that if uniformity is to be expected at all, we should naturally be induced to seek it in them; and what is more satisfactory, all gaseous bodies act similarly under similar circumstances, whether they be simple or mixed; which affords strong evidence of the truth of our position. This similarity of action is expressed by what is designated *Dalton and Gay-Lussac's law*. All gases expand equally by heat: that is, in passing from the temperature 0^0 to a^0, they expand by a value equal to $\frac{a}{100}$ times the expansion which they obtain in passing to 100^0.

Thus if V be the volume at 0^0,

V' at 100^0,

V_1 ..... a^0.

$V' = V(1 + c)$ c being some constant (the value of c appears to be $\frac{3}{8}$)

$$V_1 = V\left(1 + \frac{a}{100}c\right).$$

To find the expansion in passing from a temperature a to another $a + \beta$: let V_2 be the latter temperature,

$$V_2 = V\left\{1 + \frac{(a+\beta)}{100}c\right\}$$

$$= V_1 \frac{1 + \frac{a+\beta}{100}c}{1 + \frac{ac}{100}} = V_1\left(1 + \frac{\beta c}{100}\right),$$

if a be not very large.

The above law is independent of the pressure, provided it does not change during the change of temperature. It was enunciated from experiments made by Dalton and Gay-Lussac at moderate temperatures, but MM. Dulong and Petit have verified it for high temperatures.

88. One of the most simple and important of the facts relative to elastic media is that the pressure varies *cæteris paribus* as the density.

This may be readily conceived if we suppose the effect of doubling the density of a gas to be the same as doubling each particle of it, and then dismiss the idea of an increased action in an individual particle, owing to the increased number of particles. It is called Mariotte's Law. From this law combined with Dalton's, it follows that the elastic force at

a temperature t is equal to $K\rho'\left(1+\frac{ct}{100}\right)$, where K is a constant factor depending on the nature of the gas, ρ' the density, and $c=\frac{3}{8}$.

For if V, ρ be the volume and density at 0^0,

V' = volume at t^0,

$V'=V\left(1+\frac{ct}{100}\right)$ if the pressure remain constant,

but $V\rho=V'\rho'$;

$$\therefore\ \rho=\rho'\left(1+\frac{ct}{100}\right).$$

Now apply a pressure such that the volume becomes the same as before; let p' be this pressure, p the original ditto;

then since the pressure varies as the density,

$$p' : p :: \rho : \rho',$$

$$p'=\frac{p\rho}{\rho'}=p\left(1+\frac{ct}{100}\right);$$

and $p=K\rho'$

$$\therefore\ p'=K\rho'\left(1+\frac{ct}{100}\right).$$

89. From the experiments and admirable theoretical views of Mr Dalton, we may safely conclude, that whatever be the nature of the action of particles of the same gas on each other, if there be *any* action, the particles of different gases exert on each other respectively no sensible action. When brought together, two gases will intermingle with each other gradually, offering no other resistance to the mixture than the opposition which pebbles present to the

motion of water. It is true, the rapidity of mixing is dependent on the relative densities of the gases, and appears from Mr Graham's experiments, to be such that the mass which a gas *A* in any small time communicates to the gas *B*, is to the mass of *B* which it receives, as the square root of the density of *A* to the square root of the density of *B*.

With respect to the nature of gases, Dalton lays down the following axiom, viz.:

That every species of pure elastic fluid has its particles globular and all of the same size; but that no two species agree in the size of their particles; the pressure and temperature being the same.

It appears rather too hasty a conclusion, that because gases mix amongst each other, they therefore exert no elastic force on each others' particles. We do not intend in this part of the work to discuss its merits, but it is requisite, lest we be misunderstood, to state distinctly, that by adopting Mr Dalton's language, we must not be supposed in this individual case to attach any other signification to it than this: that it expresses fully the relation of the particles of gases in respect to their producing a joint pressure, and uniting amongst each other. It must be also observed, that as far as the former part of the reasoning is concerned, we have no right to assume that particles of the same gas are repulsive to each other.

We ought to state that Mr Dalton himself is willing to abandon the idea that the particles are mutually inelastic, which he thinks unnecessary to the explanation.

One thing is evident, that if the law be true, the most simple mode of accounting for it, and one which is not altogether fanciful, is to suppose that the *particles of the gas* produce the pressure, they being themselves acted on by caloric in such a manner that each particle is chiefly affected by a nucleus collected about itself. On this hypothesis the law is obvious.

We apply it to the following problems.

90. Prob. Gases of different elasticities contained in two vessels mix amongst each other: find the elastic force in the mixture.

Let p, V be the unit of pressure, and volume of one gas,

p', V' ..... of the other.

Then $V + V'$ is the volume of the compound.

Now the first gas when extended into the volume $V + V'$ gives a pressure

$$= \frac{pV}{V+V'} \text{ supposing it unaffected by the other.}$$

Similarly the second gives a pressure $\frac{p'V'}{V+V'}$;

hence on the hypothesis the unit of pressure

$$P = \frac{pV + p'V'}{V + V'}.$$

Cor. If $p = p'$, $P = p$,

if $V = V'$, $P = \frac{1}{2}(p + p')$.

91. Prob. To find the change of density of a gas when both its pressure and temperature are changed.

Let ρ be the density at a pressure p and temperature t,

ρ' ..... p' ..... t',

V, V' the respective volumes,

then supposing the pressure p to remain unaltered, the volume becomes $U = V\left\{1 + c\frac{(t'-t)}{100}\right\}$.

Now this volume is changed to V' by a pressure p';

$$\therefore \frac{p'}{p} = \frac{U}{V'} = \frac{V}{V'}\left(1 + c\frac{t'-t}{100}\right)$$

$$= \frac{\rho'}{\rho}\left(1 + c\frac{t'-t}{100}\right);$$

$$\text{but } p = K\rho\left(1 + \frac{ct}{100}\right);$$

$$\therefore p' = K\rho'\left(1 + c\frac{t'}{100}\right),$$

$$\text{and } \rho' = \frac{p'}{K} \cdot \frac{1}{1 + \frac{ct'}{100}}, \text{ the density required.}$$

Cor. If we take K such that $p = K\rho$, we shall have

$$p' = K\rho'\left(1 + c\frac{t'-t}{100}\right); \qquad \therefore \rho' = \frac{p'}{K\left(1 + c\frac{t'-t}{100}\right)},$$

which is the more usual formula.

92. Prob. Gas combined with moisture is confined over mercury, to determine the volume.

We shall see hereafter that vapour has the same properties as gases, provided the tension on it is less than that due to its maximum elasticity at the existing temperature.

Let V be the volume of the gas, p its unit of pressure, V' its volume when combined with vapour, the elastic force of which is f.

Then the elastic force of the gas in its mixed state $= p \cdot \frac{V}{V'}$, but the whole pressure is still p caused by the mercury;

$$\therefore\ p = \frac{pV}{V'} + f;$$

hence $V' = \dfrac{pV}{p - f}$ is the volume required.

93. Prob. When a given mass of air is suddenly compressed or dilated, to find the change of pressure.

By compression the temperature is increased, and the law of its increase is that it is proportional to the diminution of volume.

Retaining then the usual system of notation, let p_0, ρ_0, V_0 be the pressure, density, and volume at 0^0,

$$p,\ \rho,\ V \text{ at } t^0,$$

$$p',\ \rho',\ V' \text{ at } t'^0,$$

$$\text{then } p = K\rho\left(1 + \frac{ct}{100}\right),$$

$$p' = K\rho'\left(1 + \frac{ct'}{100}\right)$$

But $t' - t$ varies as $V - V'$,

which varies as $\dfrac{1}{\rho} - \dfrac{1}{\rho'}$.

Let $\therefore\ t' - t = a\dfrac{\rho' - \rho}{\rho\rho'}$;

$$\therefore\ p' = K\rho'\left(1 + \frac{ct}{100} + \frac{ca}{100}\cdot\frac{\rho' - \rho}{\rho\rho'}\right).$$

Now if the change of volume, and consequently of temperature, be very small, put

$$\rho' = \rho + \delta\rho, \qquad p' = p + \frac{dp}{d\rho}\delta\rho + \ldots\ldots;$$

$$\therefore\ p + \frac{dp}{d\rho}\delta\rho + \ldots\ldots = K(\rho + \delta\rho)\left(1 + \frac{ct}{100} + \frac{ca}{100}\cdot\frac{\delta\rho}{\rho^2}\right) + \ldots\ldots$$

$$= K\rho\left(1 + \frac{ct}{100}\right) + K\delta\rho\left(1 + \frac{ct}{100} + \frac{ca}{100}\cdot\frac{1}{\rho}\right)$$

$$= p + K\delta\rho\left(1 + \frac{ct}{100}\right)\left\{1 + \frac{\frac{ca}{100}}{\rho\left(1 + \frac{ct}{100}\right)}\right\} + \ldots$$

but $\rho\left(1 + \frac{ct}{100}\right) = \rho_0 \ldots\ldots\ldots (88)$;

$$\therefore\ \frac{1}{p}\cdot\frac{dp}{d\rho} = \left(1 + \frac{ca}{100\rho_0}\right)\frac{1}{\rho},$$

a quantity independent of t; hence if the factor of ρ be called k,

$$\log p = k \log \rho + C,$$

$$\log \pi = k \log R + C,$$

$$\log \pi' = k \log R' + C;$$

$$\therefore\ \log \frac{\pi'}{\pi} = k \log \frac{R'}{R},$$

$$\text{and } \frac{\pi'}{\pi} = \left(\frac{R'}{R}\right)^k,$$

where π, R represent the pressure and density at temperature T,

$$\pi',\ R' \text{ at } T'.$$

The value of k is 1.4161.

Cor. $\pi = K\rho\left(1 + \frac{cT}{100}\right)$;

$$\therefore \frac{\rho'}{\rho} \cdot \frac{1 + \frac{cT'}{100}}{1 + \frac{cT}{100}} = \left(\frac{\rho'}{\rho}\right)^k,$$

$$\text{and} \quad \frac{1 + \frac{cT'}{100}}{1 + \frac{cT}{100}} = \left(\frac{\rho'}{\rho}\right)^{k-1},$$

94. PROB. We stated (14) that the specific heat of a gas may be regarded under two different points of view: (1) when the pressure is constant; (2) when the volume is so: we proceed to find the relation between these quantities.

First, let the pressure remain constant, and let a quantity of heat raise the temperature by t, and cause the density to become ρ';

$$\therefore p = K\rho'\left(1 + \frac{c}{100}t\right);$$

but $p = K\rho$.

Now suppose this compressed to its original density, it will have an increment of temperature

$$t' - t = a\frac{\rho - \rho'}{\rho\rho'} = \frac{aK}{p} \cdot \frac{ct}{100};$$

hence its total increase of temperature is

$$\left(1 + \frac{aKc}{p100}\right)t = \left(1 + \frac{ac}{\rho 100}\right)t.$$

Now the increase of temperature due to a certain given quantity of heat varies inversely as the specific heat, hence

$$\frac{\text{specific heat when the pressure is constant}}{\text{..... volume .....}} = 1 + \frac{ac}{\rho 100}$$

$$= k \quad (93).$$

95. Prob. To find the difference of temperature of two stations in terms of the barometric pressures at the stations.

Let b be the column of mercury sustained by the air at the lower station.

β at the upper,

x the altitude;

then the usual formula is

$$dp = \rho g dx$$

$$= g \frac{p}{K} dx;$$

$$\therefore \log p = \frac{g}{K} x + C$$

$$\log \frac{b}{\beta} = \frac{g}{K} x;$$

$$\frac{b}{\beta} = e^{\frac{g}{K} x}$$

$$= 1 + \frac{g}{K} x + \frac{g^2}{2K^2 x^2} + \ldots$$

$$\frac{\beta}{b} = 1 - \frac{g}{K} x + \frac{g^2}{2K^2 x^2} - \ldots$$

$$\therefore \frac{b}{\beta} - \frac{\beta}{b} = 2 \frac{g}{K} x \text{ nearly.}$$

Now every 90 fathoms of ascent diminishes the temperature by about 1^0;

$\therefore x = 90 . t$ in fathoms, if t be the temperature.

But experiment gives $\frac{K}{g} = .4343 \times 10058$;

$$\therefore\ x = \tfrac{1}{2}\frac{K}{g}\left(\frac{b}{\beta} - \frac{\beta}{b}\right);$$

$$\text{and } t = \frac{1}{180}\frac{K}{g}\left(\frac{b}{\beta} - \frac{\beta}{b}\right)$$

$$= 24\tfrac{1}{4}\left(\frac{b}{\beta} - \frac{\beta}{b}\right)$$

$$= 25\left(\frac{b}{\beta} - \frac{\beta}{b}\right),$$

which is Leslie's formula.

96. The formation of gas or vapour is effected by the application of heat: in the same manner the gas may be made liquid by the application of cold, and also by pressure, but gases retain their consistence more obstinately than vapour.

The subject of vaporization is one which has occupied considerable attention, as it is continually going on at the surface of the earth, and producing effects the most important to the conservation of the state necessary to its action on life.

Vapour is not produced in the same quantity from all substances subjected to the action of the same agents; it appears that different bodies are endued with different powers of cohesion, and when that cohesion is small they are readily vaporized. It is true there are some bodies which resist all the efforts that can be made to convert them into vapour, and are consequently termed *fixed*, but we conceive that were it in our power to present a sufficiently intense heat they would be proved *volatile*.

In general fluids are more readily converted into vapour than solids, and, for the most part, the solids become fluid previous to their passing into vapour.

The quantity of vapour formed from a given bulk of matter depends entirely on the nature of that matter: thus, water expands to 1696 times its volume, whilst ether expands to only 443 times the same; the pressure being 29.92 in. and the temperature 100° in both cases.

97. Vaporization is taking place at the surfaces of fluids at all times and under most circumstances. When it proceeds slowly at low temperatures, the effect is termed evaporation; but by the application of great heat a more rapid mode of producing vapour is effected. The point at which it commences to be formed so rapidly as to produce *ebullition* is termed the boiling point, and under the pressure of an atmosphere equal to 29.9 in. of mercury it is 100° of the thermometric scale. If the pressure be diminished, the requisite temperature will be also diminished. Thus, water boils in *vacuo* at a temperature 22°, and ether at a temperature 42° below zero.

Water cannot be heated beyond the boiling point except it be confined in a close vessel so as to prevent the vapour from flying off. By confining it we can produce any degree of temperature in the water which we please. In so doing the steam generated exerts a great pressure on the sides of the vessel arising from the large quantity formed. The steam itself does not increase in elasticity by an increase of temperature; a given quantity of it contained in a vessel without water, may be heated to any extent without danger of bursting. The elastic force of steam at the boiling point is that of the atmosphere, whilst at the temperature 215° it is twenty times as great.

The ratio of increase is more rapid at high temperatures than at low ones: at 172° the pressure is eight times that of the atmosphere, and it requires an elevation of 5° to

render it nine: at 215° it only requires $2\frac{1}{2}$° additional temperature to raise the pressure to twenty-one times that of the atmosphere. We need hardly remark the advantage accruing from this circumstance in the application of steam as a means of producing motion.

98. *Evaporation* differs from ebullition in the slowness with which it proceeds, and in its forming only at the surface of the fluid.

It depends on the nature of the fluid, and experiment shews that fluids which boil at the lower temperatures evaporate the most readily. Since evaporation goes on only at the surface, it is obvious that the quantity of vapour produced will vary as the extent of surface.

The most important variation in the rate of formation of vapour, is that due to the temperature. At a high temperature the rate is greater than at a low. The quantity of vapour already in the atmosphere of course will tend to retard this action. In fact the elasticity of the vapour due to the temperature, above that of the vapour in the atmosphere, determines the rate of evaporation.

If the air be in motion, the moisture formed is immediately borne away, and thus that formed in the next instant will be as great or nearly so as that in the first.

When the atmosphere is removed from the surface of water, evaporation goes on more readily than before, but it does not appear that this is occasioned by the removal of the pressure as an elastic force, for if a vessel have a little water put into it, the quantity of vapour which will be formed is the same whether the vessel be filled with air or be a vacuum.

We have stated before that a large quantity of heat is absorbed by evaporation; so great is the quantity that Leslie invented a contrivance for freezing water by its own evaporation.

Advantage is taken of this circumstance in the East to procure water at a temperature lower than that of the air. The water is placed in porous vessels made of earth termed *alcarazas*, which as it were, present a surface to the air on every side, and thus evaporation goes on rapidly, and the water cools.

In a similar manner the air in apartments in the East Indies is cooled by placing at the doors branches of trees kept damp by being sprinkled with water.

99. It appears from the experiments of Dalton, that there is a certain density to which the vapour tends by the formation of fresh quantities, which depends on the temperature.

When the vapour has attained this state, which is its state of maximum elasticity corresponding to the particular temperature, vaporization ceases; and if the volume of the vapour in this state be diminished, a portion of the vapour liquefies until it has acquired the same elasticity it had before. Since this maximum of elasticity is greater at higher temperatures, it is obvious that a diminution of temperature will also cause some of the vapour to liquefy.

The same is true of vapour from other substances, but the particular maximum elastic force of any vapour at a given temperature depends on the nature of the vaporized substance: for instance, the elastic force of vapour of water at temperature $40\frac{5}{9}^0$ is 2.18 in. according to Dalton, 2.10 in. according to Ure; whilst that of ether is 30 in. at the same temperature.

Dalton came to the conclusion that the tension of all vapours is the same at equal distances from their temperature of ebullition, provided the distance is not extremely large; thus, water boils at 100^0, ether at 40^0, alcohol (specific gravity = .813) at 78^0; at 28^0 below the boiling point respectively, their elastic forces are 9.6, 10.3, 9.3, and at 28^0 above, they are 68.7, 73.6 and 78.5, which nearly but not exactly satisfy the law.

100. With the subject of evaporation is intimately connected that of the formation of dew, rain &c. It is not our intention to enter at any length into this subject, referring it to treatises on Meteorology. We have seen that the tension of vapour depends entirely on the temperature, and that a diminution of temperature causes a reconversion of the vapour to water. It is obvious then that a quantity of moisture will be deposited whenever a sufficient diminution of temperature takes place. Now evaporation will go on rapidly during the time of increase of heat, and will acquire such elasticity that its maximum may be considerably above that due to the temperature of the night, thus will there be a deposition of dew, the quantity of which will be greater the greater has been the evaporation during the day, and the greater the change of temperature from day to night. A point of temperature of considerable importance is the *dew point*, from the circumstance that it enables us to ascertain the exact state of the vapour in the atmosphere: for the temperature at which dew first begins to form, is that corresponding to the elastic force of the vapour actually existing in the atmosphere, and we have seen that this tension measures the quantity of vapour in a given quantity of the atmospheric air.

101. Many instruments termed hygrometers have been invented for the determination of the point in question. We shall be content with the description of one. It consists of a delicate mercurial thermometer, the bulb of which is made of thin black glass, and, with the exception of about one fourth of the surface, is covered with muslin. On moistening the muslin with ether, a diminution of temperature takes place and the mercury falls. Presently the uncovered portion of the bulb becomes dim by the deposition of moisture: if the altitude of the mercury at the instant when this is first perceptible be observed, it will be a little below the temperature of the dew point.

Of course this contrivance is somewhat rude, for the exact instant at which deposition is noticed will depend considerably on the state of the light, and moreover, we cannot be sure that the uncovered part of the bulb has acquired the same temperature as the mercury. By removing the muslin and noting the temperature at which the deposit has disappeared, which is *above* the dew point, a more accurate determination may be effected. For the point in question will be nearly a mean between the two observations.

We do not conceive that the discussion of the theory of the formation of clouds and the production of rain, hail, snow, &c. falls within the limits of our subject, and shall therefore refer those to works on Meteorology who desire information on that head.

CHAPTER V.

POLARIZATION, &c.

102. The analogies which subsist between heat and light, are not in general of a nature so striking as to compel us to adopt the same hypothesis for both. It is true the heat which accompanies solar light appears to possess properties in common with the light. It travels with the same velocity, is turned aside by the same substances, and wherever the light manifests itself, heat appears along with it. But there is no necessity that we should on this account draw the inference that light and heat are the same thing under a different representation to the senses. We want only an exciting cause, and doubtless, light is a very probable one.

M. Malus, to whom we are indebted for many valuable discoveries in the science of light, directed his attention to the discovery of analogous properties in heat. Polarization of heat accompanying solar light was distinctly established by him.

103. Heat, however, from terrestrial sources presented difficulties to the investigation which appear to have arrested its progress at a very early stage. It is true, M. Berard affirmed that he had proved polarization to exist from such sources, but we have no measures by which to judge of the fact, and succeeding philosophers till very lately doubted its reality. M. Berard considered he had ascertained polarization by reflexion, which experiment has since been carefully repeated by Professor Powell without a satisfactory result. The nature of the instrument used to

measure the heat operated as the barrier to their success. The invention of the thermo-multiplier described in Art. 9, opened a new road to the field of discovery, and by its means we are put in possession of analogies between heat and light more clearly drawn than could have been reasonably anticipated.

By an admirable series of experiments M. Melloni and Professor Forbes have succeeded in tracing the most intimate connexion between light and terrestrial and even non-luminous heat, a connexion not confined to its obvious and simple properties, but kept up throughout its more extraordinary developments, which have only been presented to notice by torturing nature and establishing circumstances of action totally different from those under which it is commonly manifested. From such a connexion the idea of a mere exciting cause residing in the light appeared to be untenable, and it wanted little to make it appear that light and heat are the same agents, the mode in which they affect the senses depending on different parts of the operation. Thus, if light be supposed to consist of the vibrations of a subtle fluid which is diffused throughout the substance of bodies, we might conceive the mechanical effect of the action of the particles, on each other, to constitute light, whilst the action on the material particles of bodies might be supposed to constitute heat. To discuss the nature of the hypothesis to be adopted will form the subject of a subsequent chapter; we are concerned at present with tracing the laws from experiment.

104. The first point established is, that the cause of the small transmissibility of dark heat through glass screens arises not from the nature of the heat so much as from that of the screens; in other words, we may find screens which will transmit all heat equally—a screen of rock salt of 2 millimètres thickness transmits, according to Melloni, $\frac{92}{100}$ of the rays incident either from flame or from boiling

water. Whence the same author remarks, that rock salt bears the same relation to heat which glass does to light.

Not only can substances be found which do not refuse to transmit non-luminous heat, but others which, though they absorb the light yet permit a portion of the heat to radiate through them. The substances experimented on were black glass and black mica: it was found that a plate of glass two millimètres in thickness transmitted $\frac{16}{100}$ of the rays incident from a lamp, or from incandescent platinum; $\frac{8}{100}$ from brass heated to 390°, and none from brass heated to 100°.

A very singular circumstance connected with this subject is the fact, that by placing two plates of tourmaline in such a manner as to intercept nearly all the incident light, Professor Forbes found that $\frac{84}{100}$ of the incident rays of heat were transmitted. After heat has passed through one screen of glass, it has less difficulty in passing through a second and so on: whence, it would seem that the rays which pass, have a particular propensity for transmission, and are not of the same nature as the intercepted rays.

105. The next thing which Melloni establishes is that non-luminous heat is susceptible of *refraction*. It appears that the index of refraction for heat by rock salt is nearly the same as for light, and moreover that there is a corresponding power of dispersion, the latter being less than for light.

106. *Polarization*. In order to understand the nature of what is termed polarization of heat, it will be requisite to explain that of polarization of light. The most simple form in which it can be exhibited is this. Let light coming from the clouds fall at a moderate angle of inci-

dence, say 50°, on a plate of unsilvered glass, or on any polished surface not metallic, as a smooth table for instance. Let the rays reflected from this substance be received on another smooth surface, such that the angle of incidence shall be about the same as before, then it is found that the quantity of light reflected from the second surface depends essentially on the position of its plane with respect to that of the first. When the reflexions take place both in the same plane perpendicular to the line of intersection of the two reflectors, a considerable quantity is reflected; when, on the contrary the surfaces are so inclined that the plane in which the second reflexion takes place is perpendicular to that of the first, very little light is reflected. This, though the most simple mode of performing the experiment, is not the best for representing it to the mind. Suppose two plates of tourmaline cut into slices in such a manner that the axis of the crystal is parallel to the surfaces of the slices. If the eye be placed so as to look at the light of the heavens through one of the slices, it will perceive no difference between the light coming through the crystal and common light, but if both slices be placed together and looked through, it will be found that when they are in the same positions with respect to each other as when in the uncut crystal, a large quantity of light passes through them: when in a different position less light passes, and when a line on the surface of one which originally was parallel to a certain line on the surface of the other is placed perpendicularly to it, almost all the light is lost, and thus, two substances, each of which is like a bit of coloured glass, will, when placed together in this manner, stop the light as though they were opaque.

107. In order to represent these phenomena the more readily, as well as to convey an idea of the mode in which they are accounted for, we will briefly explain the undulatory theory. Light is supposed to consist of the vibrations of a fluid which pervades the substance of all bodies and is designated ether. These vibrations are supposed to be

transverse to the direction of transmission; thus, if *A* be a luminous object seen by an eye at *B*, we suppose that the impression is conveyed to *B*, by means of the vibrations of the particles of ether lying between *A* and *B*, and the motion of any individual particle is in a plane perpendicular to the line *AB*. An idea of this can be conveyed to the mind by means of the motion of a rope, which if held by two persons at *A* and *B*, would communicate to *B* a jerk given to it at *A*. But the motion of any particle in the rope if held horizontally is evidently a vertical motion; in fact a *wave* passes along the rope from *A* to *B*.

Another method of representation is that of the waves in a sheet of still water as a canal. If a stone be thrown in at *A*, the eye will be caught by the motion of a wave in the direction of *B*, and the disturbance will in time manifest itself at that point; but a more accurate observation of any substance in the water, such as a bit of cork or of froth floating in or on it, will shew that no motion has taken place from *A* towards *B*, but that every particle has been made to oscillate up and down, and the transfer has been a *state of motion* which the particles more distant from *A* have acquired at greater intervals.

108. When, however, we would desire to obtain a more accurate conception of what takes place in the transmission of light from the point *A* to the point *B*, we must consider that at any intermediate point *P* the ether has been disturbed on the side next *A*; hence the forces by which *P* was retained at rest have been altered, and *P* consequently starts into motion. Then, motion having commenced, it remains to consider the forces which act upon it. In order to do this, if we suppose the whole mass in a state of motion, it is proved that the force acting on any particle tends to bring it back to its position of equilibrium; but as it reaches that position with a certain velocity, it will pass it on the other side of the line *AB*, and will thus continue to oscillate so long as the exciting

cause which keeps the *first* particles in their state of motion continues to act. Now it is supposed that common light is of such a nature that these vibrations take place successively on all sides of AB (not affecting regularity), so that the particle P is urged by forces α to pull it downward at time o; $-\alpha$ at time t; β to pull it in a direction making an angle with the vertical, but still in the plane perpendicular to AB at the time τ; $-\beta$ at the time $t+\tau$; and so on.

109. We are now possessed of the data requisite for the explanation of polarization.

Suppose the plate of tourmaline held to the eye which is looking through it at the light in a horizontal direction. Conceive a vertical line drawn on the surface of the plate of tourmaline parallel to the axis of the crystal; then the property of the crystal is this, that it allows the forces which are put in play by the motion of a particle of ether at its surface in a horizontal direction, to have their full effect (or nearly so), so that all the vibrations due to forces of that kind take place in the ether within the crystal just as well (popularly speaking) as in that without it; whilst, on the contrary, the tourmaline suppresses all the forces which act in a vertical direction which we call α, so that on arriving at the surface of the crystal they produce no effect on the particles within the crystal. We may illustrate this by supposing the particles of ether to be like beads strung loosely on horizontal strings, parallel to the surface of the slice of tourmaline, which if struck by others in a horizontal direction will immediately move, but will be unaffected by a vertical blow.

This illustration seems also to explain the fact, that a motion of the external particles in an oblique direction, will be only so far effective as is due to its resolved part in the direction of the length of the string, the other parts being destroyed by the constitution of the crystal.

110. Suppose now the two slices of crystal to be placed in such a manner that their axes are parallel to each other, it is clear that the identical vibrations which the former has suffered to pass are precisely those which will pass through the latter in this position. We should expect then to find nearly as many rays emerge from the two crystals thus united as from the former crystal alone, which is the fact.

Again, suppose the two crystals so united that their axes are perpendicular to each other, then the rays which emerge from the first are rays whose vibration is entirely horizontal; these fall on the second in this state, but the axis of the second crystal is horizontal, hence its property is to destroy all horizontal vibrations (or nearly all); the consequence is that those rays which the first crystal does not destroy, the second does, and the two crystals thus united oppose as effectual a barrier to the passage of light as an opaque body would do.

A precisely analogous explanation applies to the case of reflexion from glass. The rays due to motion parallel to the surface of the glass are reflected freely, whilst the others are partially lost, or refracted, which amounts to the same thing in the present consideration.

111. The plane passing through the axis of the tourmaline and the direction of the incident light is called the *principal plane of the crystal.*

The emergent light consisting of vibrations perpendicular to this plane is said to be polarized in the principal plane of the crystal.

In the same manner, in the case of reflexion from glass, the light consisting of vibrations perpendicular to the plane through the direction of the incident light and the line perpendicular to the surface of the glass, is said to be polarized in the plane of incidence, which the above plane is designated.

112. For the purpose of explanation we have selected that particular case in which the parts parallel to one plane are totally destroyed. This however is not necessary. The appearance presented by a pencil transmitted through Iceland spar is too common to need particular description; a single ray is doubled by the crystal.

The form of a crystal of Iceland spar is a rhombohedron, differing from a cube in having one of the lines joining two opposite corners, very much shorter than the others. A rhombohedron is bounded by six similar and equal rhombic faces. The two points at which three obtuse angles meet to form a solid angle being joined, any line parallel to the joining line is called the axis of the crystal. A more simple mode of conceiving the axis is to define it as a line which is equally inclined to the three edges which meet in the obtuse solid angle of the rhombohedron. Any plane passing through an axis of the crystal is called a principal plane of the crystal. That to which reference is usually made is the principal plane through the direction of incidence.

Suppose a ray to pass into the crystal in any direction: the light on entering consists of vibrations in all directions in a plane perpendicular to the direction of incidence. The property of the crystal is this, that the velocity with which a vibration perpendicular to the principal plane is transmitted is different from that with which a vibration is transmitted which takes place *in* the same plane. Now the *direction* taken by the light depends on its velocity of transmission within the crystal; the consequence is, that the two parts of the vibration are transmitted in different directions; and thus exhibit to the eye all the effects of two distinct rays.

This is termed *Double Refraction.* Each of these rays consists of vibrations perpendicular to one plane respectively; it is, therefore, essentially different from common light. Thus another rhomb of spar will not in all cases decompose it

into two; as for instance, when the second rhomb is placed on the first in a position similar to its own. But the examination of these cases belongs to the subject of Light. We shall presently give the analytical expressions for the velocities, which will make the subject more readily applicable.

113. *Depolarization.* If a crystal of Iceland spar be cut into a plate whose surfaces are perpendicular to the axis, and light be transmitted first through a tourmaline, then through this plate, and finally through a second tourmaline whose axis is perpendicular to that of the former, we are no longer presented with the darkness which we had before the introduction of the spar. We find a horizontal and vertical black bar, or rather brush, and the remaining space is occupied with rings concentric with the point of intersection of the brushes, and beautifully coloured. It appears therefore that the light which was originally made incapable of passing through the second tourmaline, has by the intervention of the spar, re-acquired the capability; or in other words, the vibrations which on emerging from the first tourmaline, were altogether horizontal, have by means of the spar become vertical in many parts.

114. We shall be able to perceive the reason of this when we recollect that the eye placed at the surface of the second tourmaline receives light from the spar in various *directions*.

Suppose we first consider the action on the light which, proceeding from different points in the first tourmaline, falls on the spar at points lying in a vertical plane through the eye perpendicular to the spar. It is obvious according to our previous definition that all the motion of vibration in this light incident on the spar is perpendicular to a principal plane of the crystal; for the plane of incidence is evidently a principal plane. The spar will consequently not divide a ray into two, hence the light will emerge from the crystal precisely as it entered, and will fall on the

second tourmaline in a manner which renders it incapable of transmission. A similar explanation applies to all points in a horizontal plane through the eye and perpendicular to the crystal.

But if we examine the light which falls in any other plane through the eye perpendicular to the crystal, we find quite a different case. The incident rays being those whose vibrations are in the horizontal plane, we may resolve these vibrations into two sets, one in the plane of incidence, the other perpendicular to it. These two sets will give rise each respectively to a vibration after entering the crystal; but the nature of the crystal is such, that their velocities will not be equal. On emerging they will therefore *not* be in the same circumstances as they were on entering.

Suppose the one vibration has been so much more retarded than the other in passing through the crystal, that points in the two which were originally distant by half the length of a wave are brought out together.

To fix the ideas, call the rays O and E; where O signifies the ordinary ray, a term applied to that whose vibration is perpendicular to the principal plane of incidence, E the extraordinary. Let the plane of incidence under consideration be conceived to lie from the right above to the left below with respect to the eye. Suppose the vibration fixed on to be such, that O is moving *upwards* to the left; then since at incidence the actual vibration was horizontal, E would have been moving *downwards* in the plane, had the crystal not intervened.

By the hypothesis that it has been retarded by one half the length of a wave, it is actually on emerging in motion *upwards*, hence it will combine with the other a motion more or less vertical, in some circumstances actually so. There is therefore no longer a total absorption of the ray by the second tourmaline.

We need only add that different obliquities in passing through the spar will give rise to varied excesses of retardation, and will cause the emergent ray to acquire different vertical motions. And if, as in the case in question, a small difference in obliquity is sufficient to produce a difference of retardation equal to the length of a wave, the number of complete series of alternations, each of which is comprised within this limit of extra retardation, will be considerable.

Thus, we have a complete explanation of the rings, with which however we are no further concerned at present, than inasmuch as they prove a recovery of the power of transmission through the second tourmaline which the light had lost by passing through the first.

115. We can readily apply the same considerations to analogous cases, as for instance, where the polarizations are effected by reflexion at the surface of unsilvered glass, &c. instead of transmission through a tourmaline.

116. *Circular Polarization.* In the above explanation of depolarization, or as it has been more appropriately termed, dipolarization, we supposed the excess of retardation of one ray above the other to be half the length of a wave. It must not be imagined that we fixed on that particular retardation as consisting with the points of greatest light; the different points will give an infinite number of values of this retardation, and the differences will determine particular circumstances in the final image.

There is one particular value of the retardation, however, which gives very curious results, viz. that of one fourth of the length of a wave. Draw a line in the plane of the paper, and trace a wave in the same plane, the direction of which coincides with this line. From the middle point C between two of its successive intersections A, B with the line, in a plane perpendicular to that of the paper, commence another wave also running along the line. Let

the two waves be equal and similar, then the resultant arising from these two will have important properties. At the point A the motion is wholly perpendicular to the plane of the paper, having its greatest value a downwards (suppose). At C it is wholly in the plane of the paper, having the same value and inwards. At B it is wholly perpendicular to the plane and upwards, and so on.

At a point between A and C at a distance $\frac{\phi 2\pi}{\lambda}$ from A, it is a motion compounded of a motion downwards

$$= a \cos\left(\frac{2\pi\phi}{\lambda}\right),$$

and a motion outwards

$$= a \cos\left(\frac{2\pi\phi}{\lambda} \pm \frac{\pi}{2}\right);$$

hence the motion at that point is equal to a, and lies in a plane which makes with the plane of the paper an angle whose complement is $\frac{2\pi\phi}{\lambda}$.

This angle will consequently vary directly as the distance from C going through all its values in the course of one wave.

Light of this kind is termed *circularly polarized light*, from the circumstance that the different motions are equally distant from the fixed line; lying in points similar to those which constitute the thread of a corkscrew. The polarization is said to be right handed, if it throws the points into the relative situations of a corkscrew, and left handed if into those of a spiral plant, as the hop. A correct notion of the state of each particle is gained by putting the screw in motion. Light which has received this modification, will fall on the second tourmaline in the same manner, whatever be its position.

Hence no effect will be produced by turning the second tourmaline so as to bring its axis successively into different positions with respect to that of the first.

117. It appears that when light suffers total internal reflexion within glass or other refracting media, the reflected light differs in this respect from the incident, that the vibrations in the plane of reflexion and perpendicular to that plane are retarded unequally, the difference being one eighth the length of a wave.

Thus, by two total internal reflexions we place the light in the circumstances just considered, and if the originally incident light consists of vibrations whose magnitude at every instant, is the same as respects two planes at right angles to each other passing through the direction of transmission, of which one is the plane of incidence, the emergent light after two total internal reflexions will be circularly polarized. The obvious mode of obtaining light such as we have supposed necessary for incidence, is to polarize common light, and suffer it to fall on the glass in such a manner, that its plane of vibration is inclined by an angle of 45^0 to the plane of incidence.

M. Fresnel caused a rhomb to be constructed for the purpose of obtaining circularly polarized light. It consists of a parallelopiped of glass whose ends are cut off by planes parallel to each other, and perpendicular to two sides of the parallelopiped, the planes making an angle of about $57\frac{1}{2}^0$ with the axis of the parallelopiped. Light incident perpendicular to one of these oblique faces, will after suffering two total internal reflexions from the sides of the rhomb emerge perpendicular to the other face, and, by what we above stated, if the original vibrations made equal angles with the planes which constitute the sides of the rhomb, the light will on emergence be circularly polarized.

118. In other circumstances of the incident polarized light, the vibrations will not be equal in all directions,

hence the term *elliptically polarized*, has been applied to it. In the present subject we shall be content with explaining the former only.

119. We proceed now to trace the analogy so beautifully established by M. Melloni and Professor Forbes between light and heat in the cases above described.

We shall however confine our attention to the results of the latter, for reasons not necessary to be specified. The instrument used for measuring the intensity of the heat was the thermo-multiplier, already described in (9). We shall not attempt to give an account of the mode by which the instrument has been improved so as to give very accurate measures of the relative intensity of different currents of heat; a description of these will be found in the Annales de Chimie for 1831, 1833, &c., as also in the Transactions of the Royal Society of Edinburgh, Vol. 13.

120. It may be as well to observe that a slight objection may possibly arise in the minds of some persons, to the use of this instrument for the determination of the quantity of heat transmitted through a medium; founded on the probability that the *galvanic* effect arises not from the *intensity* of the heat, but from some peculiar property of the heat so transmitted.

Now it is necessary, in speaking of the properties of heat, to be well assured that the characteristic which we adopt in one case, is at least analogous to that adopted in another. That property has been stated in Art. 4, to be its force in expanding substances.

121. Now light appears to present different properties, only one of which is taken as the measure of its intensity: as for instance, a power of producing disoxygenation. If this power had been adopted as the measure of the intensity of light, we should manifestly have to change at once all our notions of absorption; thus, if a substance absorbs

the part of the spectrum including the violet and the invisible space beyond it, the light after passing through this substance, will scarcely affect muriate of silver: we should therefore have termed this substance highly absorptive; whilst a substance which actually absorbs all the spectrum, except the part in question, would be considered most free from absorption, even more free from it than one which suffers all the rays to be transmitted equally: for the red and the space beyond it possess an oxygenating power.

122. It is obvious then that the *chemical* power of light cannot be taken as the measure of its intensity. Neither may the galvanic effect of heat be a proper measure of the intensity of heat. A person might conceive that the chemical effect of light is due to the *projection forwards* of the vibrating particles; and similarly, that the galvanic effect of heat is due to the *vibration* of the repulsive particles of heat; and that as we know the particles projected forwards will not vary, in different circumstances, as the intensity of vibration, so may we suppose, on the other hand, that the magnitude of the vibrations of the particles of heat will not, except under precisely analogous circumstances, vary as the quantity of those particles communicated*.

123. *Polarization.* Locatelli's lamp being placed six inches from the pile of the thermo-multiplier, the following observations were taken of the heat transmitted through two

* Since the above remarks were sent to the press I have learnt from Professor Forbes that difficulties similar to the above had struck the French Philosophers, and that an enquiry has consequently been instituted relative to the validity of conclusions arrived at by employing the thermo-multiplier as a measure of the intensity of heat presented under different circumstances. The results are perfectly satisfactory in their favour. I regret that I have not been able to obtain a sight of the report of M. Biot in the Transactions of the Academy, but it is sufficient to know that the investigation was carried on under the scrutiny of those least likely to be deceived either by prejudice or indifference, amongst whom was M. Poisson.

tourmalines whose axes in the first case were perpendicular to each other, giving darkness, in the second parallel, giving light: the deviation of the galvanometer marks the quantity of heat transmitted.

The following were the mean deviations in different cases, the tourmalines being mounted on glass:

Dark.	Light.	Ratio.
4.5	5.2	86 : 100
5.0	6.0	83 : 100
5.2	6.0	86 : 100
5.4	6.5	83 : 100.

A number of other observations verifies these conclusions; the sources of light, being the Argand lamp, incandescent platinum, and even non-luminous brass (temperature 390°). In the last case the effect is very small.

Professor Forbes in a subsequent memoir throws a doubt on these results as to the numerical values, attributing the error to the form of the surfaces; the general fact, however, he considers perfectly established.

124. The experiments which we are next to describe belong exclusively to Professor Forbes. Two bundles of mica plates were prepared and fixed in a tube in such a manner, that each was inclined to the direction of incidence at the angle which causes transmitted *light* to be polarized: they were also arranged with their principal sections parallel.

In order to conceive the experiments readily, suppose the source of heat at one end of a line, and the thermo-multiplier at the other: conceive holes bored obliquely through the bundles, in such a manner, that when this line passes through the holes the planes are in one position parallel to each other, but inclined to the line: in this position an index pointed to zero. As one of the plates was turned about this line, the index marked the degrees through which

it moved; the moveable plate being that on which the heat first fell, which is termed the polarizing plate.

Observation. Brass heated to 390°.

		Deviation.
Analyzing plate at 0°,	Polarizing do. at 0°	6½
	90°	5¼
	180°	7
	270°	6
	360° or 0°	7¼;

hence mean at 0° = 6°.9 | mean at 90° and ..... 180° = 7, | 270° = 5.6;

the ratio is 100 : 81, hence 19 rays per cent. are polarized.

With non-luminous heat from mercury at 270° rather a greater quantity was polarized by the same plates.

125. Analogous results were obtained with an Argand lamp, the lamp of Locatelli, and with platinum rendered incandescent by alcohol. With the latter the polarization was greater than from any other source, the ratio in the two cases being as high as 100 : 54.

Even water whose temperature was below the boiling point gave the ratio 100 : 95, which is perhaps the most important result. Hence the polarization of heat by transmission through mica is established. A great difficulty arises in the case of heat of low temperature, owing to its smaller susceptibility of polarization, a circumstance analogous to light; for the less the temperature the less the refrangibility; and Sir D. Brewster has proved that mica plates fail to polarize the less refrangible rays of light so completely as the more refrangible ones.

126. *Polarization by reflexion.* In addition to the laws of polarization of common light which we have men-

tioned above, it may be as well to state the following, from the circumstance that an analogous one has been observed for heat by Professor Forbes. It is this, that if *polarized* light be incident on a reflecting surface at an angle of incidence whose tangent is the refractive index of the substance, the reflected light is completely polarized in the plane of incidence. If the angle of incidence be greater than this, the new plane of polarization lies between that of incidence and primitive polarization: if less, it lies on the other side of the plane of incidence.

The angle of polarization at the surface of mica is estimated, with dark heat at 57°, with red *light* at 59°. We have already stated that the subject of polarization by reflexion had baffled the most ingenious philosophers who had attempted to verify M. Berard's discovery. One reason for its failure is, that the substance employed to reflect the heat was glass, which for a moderate angle is a very bad reflector. Professor Forbes discovered that mica reflects very well, and consequently proceeded with experiments on that substance: the following are his results,

Argand lamp without reflector.....	.55	polarized.
Dark heat from brass at 390°.....	.61	.....
Incandescent platinum	.63	.....

It is probable that the angle used was the correct polarizing angle for the heat of incandescent platinum, whilst it was too small for the flame, and too great for the dark heat.

127. *Depolarization and Double Refraction.* It has been already stated, that a crystal of spar placed in a certain manner between two tourmalines, whose axes are at right angles to each other so modifies their effect, that light passes through them in a variety of directions.

It would involve us too deeply in the subject of light, were we to attempt a complete explanation of the analogous phenomena in mica; the appearance depends now entirely on the position in which the mica plate is held, its plane being always parallel to those of the tourmalines. The *general* appearance is that of *two series* of dark brushes, not indeed as in spar rectilinear but hyperbolic, their axes having a common center. The points of intersection of the hyperbolas remain fixed with respect to the crystal, whatever be its position, and are called its poles; the plane through them perpendicular to the surface of the mica is called the principal plane of the crystal. To fix the ideas, let a line on the surface of the mica join these two points, then when the axes of the tourmalines are at right angles, to which case we shall generally confine ourselves, there are no longer two sets of hyperbolas; they have united into one. The eccentricity of this hyperbola depends on the position of the principal plane of the crystal; if it be such that the line joining the poles is parallel to the axis of either tourmaline, the hyperbola degenerates into two rectilinear brushes, one coinciding with and the other perpendicular to the principal plane, their point of intersection being the middle point between the poles.

In general the poles are surrounded by coloured bands, which vary in form as they are more distant from the pole: their figure is the lemniscate.

128. This is a brief statement of the *general* appearance: the effect of diminishing the thickness of the plate of crystal is to increase the diameters of the rings, &c. hence it is easily conceived that the dark brushes, when they become straight lines, fill the whole field of view, and when they are hyperbolas whose axes lie in the principal plane of the crystal, scarcely abstract any of the light through a thin plate. We may therefore state roughly, that when the principal plane of a thin crystal is parallel to either of the axes of one of the tourmalines, placed with

their axes at right angles to each other, no light is depolarized; but when that plane is inclined to the axes by 45°, a large quantity of light is depolarized and transmitted to the eye.

129. It is this property to which Professor Forbes sought an analogy in the case of heat, and with success. With dark heat (temperature 370°) the ratio of the quantity of heat transmitted in the former position to that in the latter, varied from 100 : 110 to 100 : 125, depending on the polarizing and analyzing plates which were bundles of mica.

With incandescent platinum the mean of the results was about 100 : 130. It was natural to expect that, with a very thin plate, more rays would be depolarized than were absorbed, but on trying the experiment with a plate of mica, which very nearly circularly polarized the transmitted light, it did not succeed. The tint given by this plate to polarized light, was a blue white of the first order, a colour lying very near the commencement of the scale of light, and probably still nearer the commencement of the scale of heat, in which the length of the waves appears to be greater than in light. On trying a similar experiment with a much thicker plate, which in fact, gave Newton's red of the second order, it was found that more heat was depolarized than was absorbed; thus presenting the apparent anomaly of an interposed substance increasing the heat transmitted.

That the result is due to the polarization, is obvious from the fact, that whilst the thin plate stopped only 30 out of 100 rays of common heat, the thick one stopped 65.

Various experiments were tried with different sources of heat, including water, whose temperature was below the boiling point, and all with analogous results.

130. The effects too, of changing the positions of the tourmalines, were precisely analogous to those produced on light.

In order to comprehend them, it is necessary to enter into calculations. These will be found in Airy's Mathematical Tracts. The result is this in the case in which the axis of the polarizing tourmaline is inclined by 45°, to the principal plane of the mica,

intensity of light

$$= a^2 \left\{ \cos^2 \alpha - \cos 2\alpha \sin^2 \frac{\pi I}{\lambda} \right\},$$

where α = angle between the axes of the tourmalines or the planes of polarization and analyzation,

I = the retardation of the one ray above the other in the double refraction,

λ = the length of a wave.

Let Q be the value of this when $\alpha = 0$;

$$\therefore\ Q = a^2 \cos^2 \frac{\pi I}{\lambda};$$

and let E be its value when $\alpha = \frac{\pi}{2}$, which is the case we have already examined;

$$\therefore\ E = a^2 \sin^2 \frac{\pi I}{\lambda};$$

so that $E + Q = a^2$ a constant, which is completely verified for heat.

131. *Circular Polarization.* The following observations were made with a plate which polarized a plum red tint of the first order. The value of I for light being .00001 inch.

The ratio is that of.....	a^2	:	E
Argand lamp.....	100	:	66
Incandescent platinum	100	:	47
Brass (370°).....	100	:	52
Mercury (500°).....	100	:	52.

It appears from these results that the value of I for heat was about $\frac{\lambda}{4}$, since $E = a \sin^2 \frac{\pi I}{\lambda} = \frac{a^2}{2}$.

This makes $\lambda = .00004$ for heat, which is considerably larger than for light.

The other property of circularly polarized light, viz. the unchangeability of the image by turning the analyzing plate, was completely imitated here, so that the property appears completely established. But the mode of circularly polarizing light most generally adopted, is to cause it to suffer two total internal reflexions, and this property is applicable to heat by means of the properties of rock salt, which transmits radiant heat about as copiously as glass does light.

Professor Forbes caused a prism to be made of this substance, with the angle given by calculation for light, and found that the effects were analogous.

When the plane of reflexion in the rhomb coincided with, or was perpendicular to the plane of primitive polarization, the ratio of the intensities was the same as if there had been no reflexion; when on the other hand the plane of reflexion in the rhomb was inclined by 45^0, to that of primitive polarization, the effect of turning the analyzing plate disappeared.

By taking the angle of the prism a little greater than the above, more complete coincidences were observed.

Professor Forbes states further, that he has discovered the same analogy to hold in the case of reflexion from metallic surfaces.

132. We have thus seen that a complete parallel exists between light and heat.

If, as there is every reason to believe, light is the result of vibratory motion, we are naturally driven to enquire into the possibility of applying the same considerations to heat. The subject is one of considerable difficulty, and must be deferred to another chapter.

CHAPTER VI.

TERRESTRIAL HEAT.

133. The subject of Terrestrial Heat comprises two distinct branches; the one relating to the heat far within the crust, being a discussion of its probable variation during centuries, the other referring to the rapid alteration of the external crust and portions lying at no considerable depth below the surface.

We have already, in the chapter on Conduction, solved the problem of the cooling of a sphere originally heated: as far as that part of the subject is concerned, it will remain only to add a few observations which apply the solution to the case of the earth. We will commence with the consideration of the latter case, and examine the variations which take place by the periodic changes of the position of the sun.

Of course the solution of the problem will not be considered by any means an accurate one, as indeed it is not possible to approximate very nearly to the actual state of things in the globe, depending as they do on the numberless variations in the consistence, form, density, &c. of the crust, and the ever-changing state of the atmosphere.

134. Prob. Suppose that the surface of a solid globe of very great diameter is subject to periodic changes of temperature, to find the effects of these variations at a small depth below the surface.

For r put $X-u$, so that u is the depth at which the temperature is v,

$$\text{then } \frac{d^2v}{dr^2} = \frac{d}{dr}\left(\frac{1}{\frac{dr}{dv}}\right) = -\frac{d}{dr}\frac{dv}{du}$$

$$= +\frac{d}{du}\cdot\frac{dv}{du} = \frac{d^2v}{du^2};$$

$$\therefore \text{ the equation } \frac{dv}{dt} = \frac{K}{CD}\left(\frac{d^2v}{dr^2} + \frac{2}{r}\frac{dv}{dr}\right) \quad (58) \text{ becomes}$$

$$\frac{dv}{dt} = \frac{K}{CD}\left(\frac{d^2v}{du^2} + \frac{2}{X-u}\frac{dv}{du}\right)$$

$$= \frac{K}{CD}\cdot\frac{d^2v}{du^2}, \quad \text{nearly,}$$

$$= K\frac{d^2v}{du^2}, \quad \text{if } CD = 1.$$

135. By the hypothesis the solution of this equation must, when $u = 0$, satisfy some *periodic* function of t, which represents the state of heat of the surface, varying at different times of the year.

Let this function be $\phi(t)$.

Now $\sin nt$ would satisfy the periodicity of the function, but would vanish when $t = 0$, which it ought not. But $\sin(nt - mu)$ will satisfy the result, and $v = be^{cu}\sin(nt - mu)$ is a particular integral;

$$\therefore n = K(-2cm); \quad c^2 = m^2.$$

If $c = -m$,

$$v = be^{-mu}\sin(2m^2Kt - mu),$$

and a complete solution is

$v = \Sigma e^{-mu}\{a\cos(2m^2Kt - mu) + b\sin(2m^2Kt - mu)\}$,
subject to the condition

$$\phi(t) = \Sigma(a\cos 2m^2Kt + b\sin 2m^2Kt).$$

If p be the period of the function, the equation is not altered by putting $t+p$ for t; hence

$$\phi(t) = \Sigma\{a\cos(2m^2Kt + 2m^2Kp) + b\sin(2m^2Kt + 2m^2Kp)\};$$

$$\therefore\ 2m^2Kp = 2i\pi,\quad i \text{ any integer};$$

$$\therefore\ \phi(t) = \Sigma_{i=0}^{i=\infty}\left(a_i\cos\frac{2i\pi}{p}t + b_i\sin\frac{2i\pi}{p}t\right).$$

The values of a_i, b_i are to be obtained in the usual way, by integrating $\int\phi t\sin\frac{2i\pi}{p}tdt$ and the like between the limits $t=0$ and $t=p$.

$$\text{Thus,}\quad \int_0^p \phi t\,dt = a_0 p,$$

$$\int_0^p \cos\frac{2i\pi}{p}t\,.\,\phi t = a_1\frac{p}{2},$$

$$\&c. = \&c.$$

and hence

$$v = \frac{1}{p}\int_0^p \phi t\,dt + \frac{2e^{-u\sqrt{\frac{\pi}{Kp}}}}{p}\begin{cases}\cos\left(\frac{2\pi}{p}t - \sqrt{\frac{\pi}{Kp}}u\right)\int_0^p \cos\frac{2\pi}{p}t\phi t\,dt\\ \sin\left(\frac{2\pi}{p}t - \sqrt{\frac{\pi}{Kp}}u\right)\int_0^p \sin\frac{2\pi}{p}t\phi t\,dt\end{cases}$$

$$+\frac{2}{p}e^{-u\sqrt{\frac{2\pi}{Kp}}}\begin{cases}\cos\left(\frac{4\pi}{p}t - \sqrt{\frac{2\pi}{Kp}}u\right)\int_0^p \cos\frac{4\pi}{p}t\phi t\,dt\\ \sin\left(\frac{4\pi}{p}t - \sqrt{\frac{2\pi}{Kp}}u\right)\int_0^p \sin\frac{4\pi}{p}t\phi t\,dt\end{cases}$$

$$+ \&c.\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

136. Cor. 1. By giving to u a value U we shall find that if U be tolerably large the terms of the series will be very small after the first, so that at such a depth the temperature will be constant. The principal variation of this constant temperature at any other depth not so great as U is

$$= e^{-\sqrt{\frac{\pi}{Kp}}u}\left(a_1 \cos\frac{2\pi}{p}t + b_1 \sin\frac{2\pi}{p}t\right),$$

which has its maximum when $\tan\frac{2\pi}{p}t = \frac{a_1}{b_1}$, and its value is

$$e^{-\sqrt{\frac{\pi}{Kp}}u}\frac{a_1^2 + b_1^2}{\sqrt{a_1^2 + b_1^2}} = \sqrt{a_1^2 + b_1^2}\,.\,e^{-\sqrt{\frac{\pi}{Kp}}u}\,.$$

Cor. 2. The distance to which a variation extends depends on the magnitude of $\sqrt{\frac{\pi}{Kp}}u$, so that if u, u' be the depths to which the variations are sensible due to two causes whose periods are p, p',

$$\frac{u}{u'} = \sqrt{\frac{p}{p'}}.$$

Cor. 3. If $p = 365$, $p' = 1$,

$$\frac{u}{u'} = 19.....$$

hence the yearly variation penetrates upwards of 19 times deeper than the diurnal one.

137. Cor. 4. The different points in the same vertical line will not all come to the mean temperature at the same time.

Our equation, as far as its principal terms go, may be put under the form

$$v = V + e^{-mu}\left\{a\cos\left(\frac{2\pi}{p}t - mu\right) + b\sin\left(\frac{2\pi}{p}t - mu\right)\right\},$$

or if $\tan\phi = \frac{a}{b}$,

$$v = V + \frac{be^{-mu}}{\cos\phi}\sin\left(\frac{2\pi}{p}t - mu + \phi\right)$$

$$= V + \sqrt{a^2+b^2}\,e^{-mu}\sin\left(\frac{2\pi}{p}t - mu + \phi\right);$$

hence the point whose depth is u will attain its mean temperature when $\frac{2\pi}{p}t - mu + \phi = 0$, or $= \pi$, or $= 2\pi$.....

$$\text{hence } t_0 = \frac{p}{2\pi}(mu - \phi),$$

$$t_1 = \frac{p}{2\pi}(\pi + mu - \phi),$$

$$t_1 - t_0 = \frac{p}{2};$$

also v has its greatest value when

$$t' = \frac{p}{2\pi}\left(\frac{\pi}{2} + mu - \phi\right),$$

and its greatest negative value when

$$t'' = \frac{p}{2\pi}\left(\frac{3\pi}{2} + mu - \phi\right),$$

from which it appears that the change is of the same nature as the external changes, increasing from the mean to the maximum, then diminishing to the minimum, and so on; and that the periods of the successive increases and diminutions are equal.

The maximum difference between the variable temperature and the mean is $\sqrt{a^2+b^2}\,e^{-mu}$, which quantity call A.

138. Cor. 5. A table of the results of observations made during 17 years in the observatory at Paris, at the depth of 28 mètres below the surface of the ground, will be found in M. Poisson's Theory of Heat, p. 412. It appears from that table, that supposing the thermometer itself not to have varied, (which we should think might easily be ascertained) the mean temperature is continually increasing.

Its mean from July 1817 to July 1820 is $11^{0}.73$; from July 1820 to March 1826 is $11^{0}.801$; from March 1826 to November 1828 is $11^{0}.857$; and from November 1828 to February 1835 it is $11^{0}.950$.

The appearance of the results is extraordinary, and not such as might be anticipated.

Thus in 1829, from January 1 to June 1, the value is 11.919 each month; during the month of June it is 11.989, from which time to the middle of September it is 11.954, then again 11.919 to the 1st of November, from which to the 16th it is 11.954, and finally becomes 11.919 to the end. In this case there was an increase of temperature in the summer, in other years there was a diminution. What strikes us most is that the results for a twelvemonth together, as from the middle of July 1833 to the middle of July 1834, are identically the same quantity 11.954 to three places of decimals.

This circumstance argues that the deviations (which are generally sudden ones) from an uniform rate of increase in the observations, were owing to extraneous causes. At all events, we do not know how observations could be registered to three places of decimals with such nicety, that these causes should not at least influence the third place.

A series of observations, to which we are inclined to attach importance, are those of Mr Fox, to whom we are indebted for many interesting memoirs in the *Transactions of the Royal Geological Society of Cornwall.* We shall have occasion to extract from them presently.

139. Prob. To find the total quantity of heat which at a given spot penetrates the crust of the earth during a year.

The quantity of heat which passes across a surface S is equal to $-K\frac{dv}{du}S\,dt$: also if we retain only the large term

$$-\frac{dv}{du} = me^{-mu}\left\{a\cos\left(\frac{2\pi}{p}t - mu\right) + b\sin\left(\frac{2\pi}{p}t - mu\right)\right\}$$

$$+ me^{-mu}\left\{-b\cos\left(\frac{2\pi}{p}t - mu\right) + a\sin\left(\frac{2\pi}{p}t - mu\right)\right\}$$

$$= m\sqrt{2(a^2+b^2)}\,e^{-mu}\sin\left(\frac{2\pi}{p}t - mu - \theta\right),$$

$$\text{if } \tan\theta = \frac{b-a}{a+b}.$$

Now in order to find the total quantity of heat which enters during the interval between the time at which the surface first begins to *acquire*, and that at which it ceases to do so, we must integrate the expression $-KS\frac{dv}{du}dt$ between two successive limits, which respectively make the above quantity vanish.

It is obvious that these limits are

$$t = (mu + \theta)\frac{p}{2\pi},$$

$$\text{and } t = (\pi + mu + \theta)\frac{p}{2\pi}.$$

If therefore M be the total quantity required,

$$M = m\sqrt{2(a^2+b^2)}\,e^{-mu}\frac{p}{2\pi}.2KS$$

$$= \frac{mp}{\pi}\sqrt{2(a^2+b^2)}\, e^{-mu}\, KS$$

$$= A\frac{mp}{\pi}\sqrt{2}\, KS$$

$$= A\frac{p}{\pi}\sqrt{\frac{2\pi}{Kp}}\,.\,KS;$$

restoring to the K in the denominator its value from (134)

$$= A\frac{p}{\pi}\sqrt{\frac{2\pi CD}{pK}}\,.\,KS$$

$$= A\sqrt{\frac{2\pi KCD}{\pi}}\, S.$$

140. PROB. To find at what time the interior of the earth commences to *receive* heat.

The value of $-\frac{dv}{du}$ in the interior varies as

$$m\sqrt{2(a^2+b^2)}\,.\,\sin\left(\frac{2\pi}{p}t - mu - \theta\right),$$

and that of $v = V + \sqrt{a^2+b^2}\, e^{-mu}\sin\left(\frac{2\pi}{p}t - mu + \phi\right)$.

Now the flux of heat commences when

$$t = \theta\,.\,\frac{p}{2\pi},$$

but at this time $v = V + \sqrt{a^2+b^2}\, e^{-mu}\sin(\theta+\phi)$,

and $\sin(\theta+\phi) = \cos\theta\cos\phi\,(\tan\theta + \tan\phi)$

$$= \cos\theta\cos\phi\left(\frac{b-a}{a+b} + \frac{a}{b}\right)$$

$$= \frac{a^2+b^2}{\sqrt{a^2+b^2}\sqrt{2(a^2+b^2)}}$$

$$= \frac{1}{\sqrt{2}} = \sin\frac{\pi}{4}.$$

Now the temperature is a mean when the coefficient of $\sqrt{a^2 + b^2} = 0$, if therefore t' be the interval elapsed since that period, $\frac{2\pi}{p} t' = \frac{\pi}{4}$, or the interval between the time at which the surface has acquired its mean temperature, and that at which it begins to conduct heat inwards to the mass of the earth, is the eighth part of a year. In the interval the interior of the earth being more heated than the surface, is continually parting with its heat, notwithstanding that the surface itself is increasing in temperature.

Now the value of the mean maximum temperature during the day given by Mr Daniell in his Meteorological Essays, p. 210, is 56.3, a value not materially different from that for April and October; the correct value is due to about the first half of these months respectively. Similarly the minimum value is 42.5, which coincides closely with the minimum values for April and October.

But the experiments of Mr Fox before mentioned, in the Huel Gorland mine, give the maximum value of the temperature at a considerable depth below the surface about the beginning of September, and the minimum in March.

141. In order to account for this, we must observe that the minimum value of the temperature at the surface occurs in January, whereas at a very considerable distance to which the variations are scarcely sensible, the minimum ought according to theory to occur about the end of May, or the beginning of June. For places between these, at which in fact the remaining terms of the series in (135) for v are sensible, but small, the minimum value of the temperature ought to occur between January and June, approaching more nearly to June as the depth is greater. At the depth of the Huel Gorland mine, it occurs in March. At that of the Dolcoath mine which is greater than the former, it appears that the minimum happens as it ought, rather later; for the difference .77 between the temperatures for

February and March is much greater than .54 the difference between March and April, hence the mean will lie nearer April. On the contrary in the Huel Gorland mine the difference .83 between February and March is less than .98 that between March and April, so that at a less depth the minimum is nearer February, as it ought to be.

A similar observation applies to the maximum temperature.

142. We can exhibit the coincidence of theory and observation on this subject in a still more striking manner, by referring to a series of observations taken by Mr Ferguson.

We present a list of the times of minimum temperature at the depths stated, the letter *E* being prefixed when they occur early in the month, *L* when late.

1 Foot.	2 Feet.	4 Feet.	8 Feet.
Jan. *L.*	Feb. *E.*	Feb. *L.*	Feb. *L.* in 1816.
Jan.	Feb. *L.*	Feb. very *L.*	March in 1817.

The following is a table of the time of the corresponding maximum temperature:

1 Foot.	2 Feet.	4 Feet.	8 Feet.
July	Aug. *E.*	Sept.	Sept. *L.* in 1816.
July	July *L.*	Aug. *L.*	Sept. in 1817.

results accurately agreeing with theory.

We do not think it worth while to give the results of the experiments in full, from the fact, that the former set were deduced from observations at different places, whilst the latter were made so near to the surface of the ground, that a considerable part of the result is certainly due to the state of the clouds, &c. in preventing or promoting *direct* radiation.

Prob. To find the total quantity of heat which enters a given area during the year, supposing the substance of the earth to be iron.

Let the unit of heat be that which suffices to convert a kilogramme of ice at the temperature 0, into water of the same temperature, then $K = \frac{6}{5}$, $C = \frac{1}{6}$, $D = 7800$ are known by experiment, or rather $CD = 1300$, the unit of time being one minute; and if Paris be the place of observation, $A = 10^0$, and $\theta = 60.24.36.5$;

$$\therefore M = 3856;$$

hence the heat is just such as would suffice to melt 3856 kilogrammes of ice, or the quantity of heat which enters a square yard of the surface of the earth, would melt about 7000 lbs. of ice.

143. In all that we have done, it is clear that our only restriction as to the nature of the mean temperature, and the coefficient of the variable terms, is that they are independent of the depth below the surface. It matters not what function of the position on the surface they form.

The most simple hypothesis respecting the mean value of the temperature, is that it varies as the cosine of the latitude. In fact the quantity of heat incident on a given area when the sine is in the equator varies thus, and we should be led to suppose that this would be true as a mean result for the year, provided the place do not lie within the tropics. We may remark, that the expression $81^0.5 \cos L$ has been assumed as the formula of mean temperature by Brewster.

144. We have seen that the variations of temperature extend to a very small depth below the surface; it remains to find the state of permanent temperature below.

Suppose all the points of the equator on the surface of a sphere to have acquired a common temperature, and that all the points of a small circle parallel to the former have also a common temperature different from the former.

Let x represent the distance of any point of the sphere from the plane of the equator, y its distance from the axis of rotation perpendicular to the equator, v the permanent temperature of the same point.

Let $x + \delta x$ be the distance of another point from the plane of the equator, and take an annulus of the breadth δy, then $2\pi y\, \delta y\, \delta x$ is its volume.

Now to find the change of its temperature, we must conceive it to take place by one transfer parallel to the pole, and another outwards; the former flux is

$$\frac{d}{dx}\left(K \frac{dv}{dx} 2\pi y\, \delta y\right) dx;$$

the latter,

$$\frac{d}{dy}\left(K \frac{dv}{dy} 2\pi y \delta x\right) dy;$$

hence we must have

$$y \frac{d^2v}{dx^2} + y \frac{d^2v}{dy^2} + \frac{dv}{dy} = 0;$$

$$\text{or } \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} + \frac{1}{y}\frac{dv}{dy} = 0.$$

It is further requisite in order that the temperature be permanent, that the surface, or at least, that points at such a depth as to exclude small periodical variations, should be exposed to such an external action as to satisfy this equation.

145. Prob. To obtain a solution of this equation.

Let $v = u \cos x$, which is only one form, as $u \cos mx$ would do equally well;

$$\frac{d^2v}{dx^2} = -u \cos x$$

$$= -v;$$

$$\text{hence } \frac{d^2v}{dy^2} - v + \frac{1}{y}\frac{dv}{dy} = 0.$$

Assume $v = a_0 + a_1 y + a_2 y^2 + \ldots a_K y^K + \ldots$

$$\therefore \frac{dv}{dy} = a_1 + \ldots + K a_K y^{K-1} + \ldots$$

$$\frac{d^2 v}{dy^2} = 2a_2 + \ldots K.(K-1)\, a_K y^{K-2} + \ldots$$

$$\therefore 2a_2 + \ldots K.(K-1)\, a_K y^{K-2} + \ldots - (a_0 + a_1 y + \ldots a_{K-2} y^{K-2} + \ldots)$$

$$+ \frac{a_1}{y} + 2a_2 + \ldots K a_K y^{K-2} + \ldots = 0;$$

hence $a_1 = 0$;

$$K.(K-1)\, a_K - a_{K-2} + K a_K = 0,$$

$$a_K = \frac{a_{K-2}}{K^2};$$

$$\therefore a_3 = \frac{1}{3^2} a_1 = 0,$$

$$a_5 = 0,$$

&c. = &c.

$$a_2 = \frac{1}{2^2} a_0,$$

$$a_4 = \frac{1}{4^2} a_2 = \frac{1}{2^2 . 4^2} a_0,$$

&c. = &c.

$$\text{and } v = a_0 \left(1 + \frac{y^2}{2^2} + \frac{y^4}{2^2 . 4^2} + \frac{y^6}{2^2 . 4^2 . 6^2} + \ldots\right)$$

is the solution of the equation.

146. To express this in a finite form, we observe that

$$v = a_0 \left(1 + \frac{1}{2} . \frac{y^2}{1.2} + \frac{1.3}{2.4} . \frac{y^4}{1.2.3.4} + \frac{1.3.5}{2.4.6} \frac{y^6}{1.2 \ldots 5.6} + \ldots\right).$$

$$\text{Now } \int_0^\pi \cos^n\theta\, d\theta = \cos^{n-1}\theta \sin\theta + (n-1)\int \cos^{n-2}\theta\, d\theta - (n-1)\int \cos^n\theta\, d\theta$$

$$= \frac{n-1}{n}\int \cos^{n-2}\theta\, d\theta;$$

hence if n be even,

$$\int_0^\pi \cos^n\theta\, d\theta = \frac{(n-1).(n-3)\ldots 1}{n.(n-2)\ldots 2}.\pi;$$

and if n be odd,

$$\int_0^\pi \cos^n\theta\, d\theta = 0.$$

By means of these expressions we can put the above series under the form

$$\pi v = a_0 \int_0^\pi d\theta \left(1 + \frac{y\cos\theta}{1} + \frac{y^2\cos^2\theta}{1.2} + \frac{y^3\cos^3\theta}{1.2.3} + \ldots\right)$$

$$= a_0 \int_0^\pi d\theta\, e^{y\cos\theta};$$

and the solution of our equation becomes

$$v = A\cos x \int_0^\pi d\theta\, e^{y\cos\theta}.$$

Cor. If we assume $v = u\cos mx$, we have

$$\frac{du^2}{dx^2} = -m^2 u\cos mx = -m^2 v;$$

$$\therefore \frac{d^2v}{dy^2} - m^2 v + \frac{1}{y}\frac{dv}{dy} = 0;$$

and the same assumption as before gives us

$$a_K = \frac{m^2 a_{K-2}}{K^2};$$

$$\therefore v = a_0\left(1 + \frac{m^2y^2}{2^2} + \frac{m^4y^4}{2^2.4^2} + \ldots\right)$$

$$= A\cos mx \int_0^\pi d\theta\, e^{my\cos\theta},$$

so that the complete solution is

$$v = \sum_{1}^{\infty}{}_m . A_m \cos mx \int_0^\pi d\theta\, e^{my\cos\theta}.$$

147. It appears to be perfectly established that the temperature of the earth increases as we proceed downwards. It becomes then an interesting speculation to account for this. Fourier and Laplace have attributed this circumstance to an original heat which has been communicated to the mass, and has by time reached such a state that it is of excessive intensity towards the center and diminishes towards the surface.

148. PROB. Suppose the earth has been heated uniformly, and then left to cool. All we have to do in this case, is to avail ourselves of the solution of the problem given in Art. (81), and apply observation to determine the constants.

When the time elapsed is very considerable, it may be shewn that the first term of the value of v is the principal one.

$$\text{Hence } v = \frac{2h}{K} X . \frac{\sin \epsilon \frac{x}{X}}{\frac{\epsilon x}{X}} . \frac{e^{-\frac{K}{CD}\frac{\epsilon^2}{X^2}t}}{\epsilon \operatorname{cosec} \epsilon - \cos \epsilon}$$

$$= A . \frac{\sin \frac{\epsilon x}{X}}{\frac{\epsilon x}{X}} \text{ suppose;}$$

where A is very nearly constant for a few years.

For small depths let $x = X - y$ where y is a very small quantity,

$$\text{hence } v = A . \frac{\sin \left(\epsilon - \frac{\epsilon y}{X}\right)}{\epsilon - \frac{\epsilon y}{X}}$$

$$= \frac{A}{\epsilon}\left(1 + \frac{y}{X}\right)\left(\sin\epsilon\cos\frac{\epsilon y}{X} - \cos\epsilon\sin\frac{\epsilon y}{X}\right)$$

$$= \frac{A}{\epsilon}\left(1 + \frac{y}{X}\right)\left(\sin\epsilon - \cos\epsilon\,.\,\frac{\epsilon y}{X}\right)$$

$$= \frac{A}{\epsilon}\left\{\sin\epsilon + \frac{y}{X}(\sin\epsilon - \epsilon\cos\epsilon)\right\}.$$

Now we have in our solution supposed the external temperature = 0. This is of course incorrect, we must therefore consider it constant which is sufficiently exact, as we have found that the variations of external temperature produce an effect only to a very slight depth below the surface. But the equation of internal motion will remain precisely the same as at present, if we suppose the *actual* temperature increased by a constant quantity V; and the equation of condition relative to the surface will be corrected by writing instead of u the actual temperature, its value $v + V$. The quantity of external radiation varies by Newton's law as the excess of the temperature of the body above the external temperature; and hence it varies as v, which we have supposed it to do.

From these considerations it is obvious that the actual temperature is

$$u = V + \frac{A}{\epsilon}\sin\epsilon\left(1 + \frac{1 - \epsilon\cot\epsilon}{X}y\right);$$

$$\therefore\ u = f + gy.$$

149. This conclusion has been examined by experiment, and a complete discussion of it will be found in M. Poisson's work.

By means of a large number of observations, the values of f and g obtained for a place in the neighbourhood of Geneva, 500 mètres above the level of the sea, are

$$f = 10^0.140, \qquad g = 0^0.0307,$$

it being understood that y is expressed in mètres. We have not thought it necessary to reduce the results to inches or feet, which may however be easily done since a mètre is 39.37079 inches.

150. Prob. To find the time which must elapse before the earth has cooled down to the external temperature,

$$\text{we have } u = V + A\frac{\sin\epsilon}{\epsilon}\left(1 + \frac{1 - \epsilon\cot\epsilon}{X}y\right)$$

$$= V + A\frac{\sin\epsilon}{\epsilon}(1 + h'y) \quad (79).$$

Now it appears that if the unit of linear measure be the mètre and the unit of time a year, $h' = 1.05719$ at Paris. Also from the experiments of M. Arago it appears that $g = 0^0.0281$.

But when $f = 0$, $g = 0$, hence the time elapsed will be that which makes $A = 0$.

$$\text{But } A = ae^{-\frac{K}{CD}\frac{\epsilon^2}{X^2}\cdot t}.$$

Now after the time t_1 from the present time the new value A_1 of A is

$$A_1 = ae^{-\frac{K}{CD}\frac{\epsilon^2}{X^2}(t+t_1)}$$

$$= Ae^{-\frac{K}{CD}\cdot\frac{\epsilon^2}{X^2}\cdot t}$$

$$= Ae^{-\frac{0.6378}{100,000,000,000}t},$$

from which we conclude that more than a thousand million centuries must elapse before A will be diminished by one half.

151. There may appear some difficulty in obtaining numerical results in this case, from the apparent want of homogeneity in the equations which give the different constants, thus ϵ is to be obtained from the equation,

$$\epsilon\cot\epsilon = 1 - h'X,$$

but since h' is in this place the ratio of the exterior conductivity to the interior, it is obvious that the unit of linear measure must affect that ratio, otherwise the value of $h'X$ would be altered by changing the unit of X. But if we remember that $\frac{1}{K}$ enters the value of h', and that K is the quantity of heat which, in the matter of the earth, would traverse an unit of area of which the extremities had the temperatures 0 and 1 respectively: we shall perceive that the flux across two planes at temperatures $12^0.1$ and $11^0.1$, whose difference of depth is 18.8 mètres (we quote actual observations) will be $\frac{K}{18.8}$; and, since it must be denoted on the same scale of temperature as that of the exterior conductivity, it is obvious that K will vary directly as the linear unit and hence $\frac{X}{K}$ which is a factor of h' will be independent of that unit. Similar remarks apply to the other constants.

152. We do not enter in the present work on the subject of the change of consistence of the earth. Obvious phenomena appear to indicate that it was originally fluid; supposing it the case, it has occupied the attention of philosophers to determine whether it ought to solidify at its surface or its center, or at both simultaneously. The subject is one in which we can arrive at no very certain results, owing to our ignorance of the compressibility of fluids at an excessive temperature and pressure. The general opinion is, that the solidification has commenced at the surface where the temperature is of course very small, but the pressure small in proportion.

Poisson is of opinion that it commenced at the center where the pressure must be enormous. Many other interesting speculations connected with this branch of the subject will be found in Poisson's work, and in the Memoirs of the Institute, and the Annales de Chimie to which we must refer the reader.

CHAPTER VII.

THEORY.

153. The branch of our subject which we are about to consider, is the application of a theoretical hypothesis to the explanation of the facts which we have already discussed.

This is doubtless a subject of considerable difficulty and uncertainty. We are not so intimately acquainted with the constitution of material bodies even in regard to their consistence, as to derive much aid from our previous knowledge. Still less are we certain that the analogies which we trace between different agents such as light and heat, are analogies which extend to the actual *identity* of the causes which conspire to produce the effects. We are left in doubt to what extent subsidiary actions connected with, but not forming a constituent part of the agency which we designate heat lead us astray in our notions, and induce us to connect together things which are essentially different. Labouring therefore under such difficulties we may be allowed considerable latitude of assumption, and may claim exemption from a rigid explanation of *every* point which we lay down. Not that we are obliged to assume connexions devoid of probability, or even such as will present to the mind more difficulty in conception, than is consistent with the supposition of their being the groundwork of an *explanation*. We are compelled it is true in some cases to leave in doubt many of the most essential features, but this doubt arises not so much from the difficulty of classifying them

and referring them to their proper head, as from that of guarding against assumptions which have no direct analogy in other cases, and which it is advisable not to introduce until subsequent observations shall make it strongly probable that they are based on evidence.

154. We exhibit at one view the principal points which we are called upon to account for, making references to the articles where they are stated and explained at length. The order in which we have placed them, will probably appear at first sight nearly the reverse of natural, but from circumstances which the reader will discover as he proceeds, it is a very convenient one.

1. Polarization (123).

2. The pressure of an elastic fluid, at a given temperature varies as the density (88).

3. All gases expand equally by heat (87).

4. If equal volumes of any two gases be taken at a given pressure and temperature, and the volume of each be suddenly changed by the same quantity, the total amount of heat developed or absorbed by each is the same.

5. Gases mix freely amongst each other (89).

6. Solids expand by heat: and in general crystals change their form.

7. A change of consistence may be produced by heat.

8. Latent heat (17).

9. The velocity of cooling is represented by $(a^{\theta} - 1)$, θ being the temperature measured from a proper point; provided the surrounding medium return no heat (40).

155. We proceed to state the hypothesis, but as it will require considerable additions as we proceed, a complete view of it will be repeated at the end.

The particles of matter in all bodies are supposed to be surrounded by particles of a subtle fluid pervading space and termed caloric, the atoms of which are so much smaller than those of matter, that each material particle may be conceived to be surrounded by a large number of them. It will be seen that thus far we have retained the same views as were adopted by Franklin for the explanation of electrical phenomena.

It is supposed that the particles of caloric repel each other, but attract those of matter, whilst the particles of matter also repel each other, the function of the distance being always the same, which we shall give reasons for assuming to be that of the inverse square.

156. Let x, y, z be the co-ordinates of any particle P of caloric when at rest, $x+\delta x$, $y+\delta y$, $z+\delta z$ those of another particle Q, $x+\Delta x$, $y+\Delta y$, $z+\Delta z$ those of a material particle M, $PQ=r$, $PM=R$, $r\phi(r)$ the law of attraction; suppose the attraction at distance unity of a particle of caloric to be P, and that of a particle of matter M.

Let $x+\alpha$, $y+\beta$, $z+\gamma$ be the co-ordinates of P at the end of the time t,

$x+\alpha+\delta x+\delta\alpha$, $y+\beta+\delta y+\delta\beta$, $z+\gamma+\delta z+\delta\gamma$ those of Q,

$x+\alpha+\Delta x+\Delta\alpha$, $y+\beta+\Delta y+\Delta\beta$, $z+\gamma+\Delta z+\Delta\gamma$ those of M,

$r+\rho$ the distance PQ,

$R+H$ the distance PM:

then $(r+\rho)^2=(\delta x+\delta\alpha)^2+(\delta y+\delta\beta)^2+(\delta z+\delta\gamma)^2$;

$$\therefore\ r^2+2r\rho+\rho^2=\delta x^2+\delta y^2+\delta z^2+2\delta x\delta\alpha+2\delta y\delta\beta+2\delta z\delta\gamma+\ldots$$

$$\therefore\ \rho=\frac{1}{r}(\delta x\delta\alpha+\delta y\delta\beta+\delta z\delta\gamma)\text{ nearly.}$$

Also the repulsion of Q on P is $P.(r+\rho)\phi(r+\rho)$, hence the resolved part of this in a direction parallel to the axis of x, is $P(\delta x+\delta\alpha)\phi(r+\rho)$. Similarly, the attraction of M on P in the same direction is $M(\Delta x+\Delta\alpha)\phi(R+H)$, hence the whole force on P tending to pull it in a direction opposite to that in which x is measured, is represented by

$$P\sigma(\delta x+\delta\alpha)\phi(r+\rho)-M\Sigma(\Delta x+\Delta\alpha)\phi(R+H),$$

$$\text{or } \frac{d^2\alpha}{dt^2}=-P\sigma(\delta x+\delta\alpha)\phi(r+\rho)+M\Sigma(\Delta x+\Delta\alpha)\phi(R+H).$$

157. To simplify this equation we will suppose that the particles of the medium are arranged in symmetry about the axes, by which means the symbol σ denoting the sum of all terms analogous to that which follows it taken throughout the medium of caloric, will embrace terms having the same value with opposite signs, and the equation will be much reduced.

We are, however, unable to decide on the terms which destroy each other, except we know previously the kind of motion amongst the particles; thus, if we suppose them to have a vibratory motion whose direction of transmission is the axis of x, we may assume

$$\alpha=a\cos(nt-kx),$$

$$\beta=b\cos(nt-kx),$$

$$\gamma=c\cos(nt-kx);$$

this hypothesis gives

$$\begin{aligned}\delta\alpha&=a\cos\{nt-k(x+\delta x)\}-a\cos(nt-kx)\\&=a\{\cos(nt-kx)\cos k\delta x+\sin(nt-kx)\sin k\delta x\}-a\\&=-a(1-\cos k\delta x)+a\sin(nt-kx)\sin k\delta x\\&=-2a\sin^2\frac{k\delta x}{2}+\frac{d\alpha}{dx}.\frac{\sin k\delta x}{k},\end{aligned}$$

$$\delta\beta = -2\beta \sin^2 \frac{k\delta x}{2} + \frac{d\beta}{dx} \cdot \frac{\sin k\delta x}{k},$$

&c. = &c.

By a similar hypothesis we may make the disturbance of a particle of matter, whose original co-ordinates are x, y, z, equal to $A\cos(Nt - Kx + B)$. Call this α';

then $$\Delta\alpha' = -2\alpha' \sin^2 \frac{K\Delta x}{2} + \frac{d\alpha'}{dx} \cdot \frac{\sin K\Delta x}{K},$$

$$\Delta\beta' = -2\beta' \sin^2 \frac{K\Delta x}{2} + \frac{d\beta'}{dx} \cdot \frac{\sin K\Delta x}{K},$$

&c. = &c.

and $$\Delta\alpha = A\cos(Nt - Kx - K\Delta x + B) - a\cos(nt - kx)$$

$$= \alpha' \cos K\Delta x + \frac{d\alpha'}{dx} \cdot \frac{\sin K\Delta x}{K} - \alpha,$$

&c. = &c.

Now $\phi(r + \rho) = \phi r + \phi' r\rho$, nearly

$$= \phi r + \frac{\phi' r}{r}(\delta x\delta\alpha + \delta y\delta\beta + \delta z\delta\gamma);$$

$$\therefore (\delta x + \delta\alpha)\phi(r + \rho) = \phi r(\delta x + \delta\alpha)$$

$$+ \frac{\phi' r}{r}(\delta x^2\delta\alpha + \delta x\delta y\delta\beta + \delta x\delta z\delta\gamma), \text{ nearly};$$

but in taking the sum of expressions similar to the above throughout the whole medium, we must remark that for every pair of values of δy, δz, there are two corresponding values of δx equal and with opposite signs; the same is true of other combinations. Also α, $\frac{d\alpha}{dx}$, &c. will be independent of the symbol σ, so that expressions such as $\sigma\alpha\delta x\delta y\delta z$, $\sigma\delta x\delta y \sin^2 \frac{k\delta x}{2}$, and the like, will vanish.

$$\text{Hence } \sigma(\delta x + \delta a)\phi(r+\rho) = -2\sigma a \phi r \sin^2\frac{k\delta x}{2}$$

$$-2\sigma a \frac{\phi' r}{r}\delta x^2 \sin^2\frac{k\delta x}{2}$$

$$= -2a\sigma\left(\phi r + \frac{\phi' r}{r}\delta x^2\right)\sin^2\frac{k\delta x}{2}.$$

$$\text{Similarly, } \Sigma(\Delta x + \Delta a)\phi(R+H) = \Sigma\left(\phi R + \frac{\phi' R}{R}\Delta x^2\right)\Delta a$$

$$+ \Sigma\frac{\phi' R}{R}(\Delta x \Delta y \Delta\beta + \Delta x \Delta z \Delta\gamma)$$

$$= a'\Sigma\left(\phi R + \frac{\phi' R}{R}\Delta x^2\right)\cos K\Delta x - a\Sigma\left(\phi R + \frac{\phi' R \Delta x^2}{R}\right);$$

$$\text{hence } \frac{d^2 a}{dt^2} = 2Pa\sigma\left(\phi r + \frac{\phi' r}{r}\delta x^2\right)\sin^2\frac{k\delta x}{2}$$

$$- Ma\Sigma\left(\phi R + \frac{\phi' R}{R}\Delta x^2\right) + Ma'\Sigma\left(\phi R + \frac{\phi' R}{R}\Delta x^2\right)\cos K\Delta x,$$

$$\frac{d^2\beta}{dt^2} = 2P\beta\sigma\left(\phi r + \frac{\phi' r}{r}\delta y^2\right)\sin^2\frac{k\delta x}{2}$$

$$- M\beta\Sigma\left(\phi R + \frac{\phi' R}{R}\Delta y^2\right) + M\beta'\Sigma\left(\phi R + \frac{\phi' R}{R}\Delta y^2\right)\cos K\Delta x.$$

158. To obtain equations for α', β'.

The above equations are those of the motion of a particle of caloric; they are consequently, *mutatis mutandis*, the equations of the motion of a particle of matter, since we have supposed the systems of particles analogous;

$$\therefore \frac{d^2 a'}{dt^2} = 2Ma'\Sigma\left(\phi R + \frac{\phi' R}{R}\Delta x^2\right)\sin^2\frac{K\Delta x}{2}$$

$$- Pa'\sigma\left(\phi r + \frac{\phi' r}{r}\delta x^2\right) + Pa\sigma\left(\phi r + \frac{\phi' r}{r}\delta x^2\right)\cos k\delta x,$$

$$\frac{d^2\beta'}{dt^2} = 2M\beta'\Sigma\left(\phi R + \frac{\phi' R}{R}\Delta y^2\right)\sin^2\frac{K\Delta x}{2}$$

$$- P\beta'\sigma\left(\phi r + \frac{\phi' r}{r}\delta y^2\right) + P\beta\sigma\left(\phi r \frac{\phi' r}{r}\delta y^2\right)\cos k\delta x.$$

159. To solve the equation.

We have already introduced the hypothesis that the solution is

$$\alpha = a\cos(nt - kx),$$

$$\beta = b\cos(nt - kx),$$

$$\alpha' = A\cos(Nt - Kx + B),$$

&c. = &c.

and it only remains to determine n, N, &c.

$$\frac{d^2\alpha}{dt^2} = -n^2\alpha,$$

$$\frac{d^2\alpha'}{dt^2} = -N^2\alpha';$$

therefore by substitution in our equations of the last Art. calling $P\left(\phi r + \frac{\phi' r}{r}\delta x^2\right)$ f, and $M\left(\phi R + \frac{\phi' R}{R}\Delta x^2\right)$ F, we obtain

$$-n^2\alpha = 2\alpha\sigma f\sin^2\frac{k\delta x}{2} - \alpha\Sigma F + \alpha'\Sigma F\cos K\Delta x,$$

$$-N^2\alpha' = 2\alpha'\Sigma F\sin^2\frac{K\Delta x}{2} - \alpha'\sigma f + \alpha\sigma f\cos k\delta x,$$

and similar relations between β and β', γ and γ'.

From the 1st, $-\left(n^2 + 2\sigma f\sin^2\frac{k\delta x}{2} - \Sigma F\right)\alpha = \Sigma F\cos K\Delta x.\alpha'$,

..... 2d, $-\left(N^2 + 2\Sigma F\sin^2\frac{K\Delta x}{2} - \sigma f\right)\alpha' = \sigma f\cos k\delta x.\alpha$;

which equations establish a relation between the quantities n and N, and give $a' = ma$.

160. We shall presently be able to pursue this investigation still further. It will be requisite first to make a considerable digression in order to determine the form of the function.

Suppose the motion to take place in a vacuum, for which we obtain the following equations:

$$\frac{d^2a}{dt^2} = 2Pa\sigma\left(\phi r + \frac{\phi' r}{r}\delta x^2\right)\sin^2\frac{k\delta x}{2},$$

$$\frac{d^2\beta}{dt^2} = 2P\beta\sigma\left(\phi r + \frac{\phi' r}{r}\delta y^2\right)\sin^2\frac{k\delta x}{2};$$

hence $\because \beta = b\cos(nt - kx)$,

$$n^2 = -2P\sigma\left(\phi r + \frac{\phi' r}{r}\delta y^2\right)\sin^2\frac{k\delta x}{2}.$$

Now it must be observed that the mode in which this result is produced is one which takes account of the *differences* of the disturbances on the two sides of the particle under consideration. The magnitude of the expression will therefore clearly depend greatly on that of the portion which arises from the particles next it. In other words $\phi(r)$ must be some inverse function of r such that it diminishes as r increases, and thus as far as this factor is concerned the particles near P will have the principal influence, but within a certain limit this will not be the case with the factor $\sin^2\frac{k\delta r}{2}$, which will increase at first until it becomes unity, and then again diminish as δx is increased. The result is obviously this, that the principal part of the term will be that which lies within the value of $\delta x = \frac{\pi}{k}$.

Now β will go through all its values whilst kx is increased by 2π, and since β is the same for $2\pi + kx$ as for kx, whatever be x, it is evident that $\frac{2\pi}{k}$ is the length of a wave λ.

Expanding $\sin^2 \frac{\pi \delta x}{\lambda}$, we obtain

$$n^2 = -2P\sigma\left(\phi r + \frac{\phi' r}{r}\delta y^2\right)\left(\frac{\pi^2}{\lambda^2}\delta x^2 - \frac{\pi^4}{3\lambda^4}\delta x^4 + \ldots\ldots\right)$$

as far as it depends on the first half-wave. Also a point $x + x'$ will have the same motion at the end of the time $t + t'$ as the point under consideration has at the time t, provided that

$$\cos(nt + nt' - kx - kx') = \cos(nt - kx),$$

$$\text{or} \quad nt' = kx', \qquad \text{or} \quad \frac{x'}{t'} = \frac{n}{k};$$

but $\frac{x'}{t'}$ measures the velocity of propagation of the motion $= v$, suppose;

$$\therefore v^2 = \frac{n^2}{k^2} = \frac{n^2\lambda^2}{4\pi^2} = -\tfrac{1}{2}P\sigma\left(\phi r + \frac{\phi' r}{r}\delta y^2\right)\left(\delta x^2 - \frac{\pi^2 \delta x^4}{3\lambda^2}\right).$$

Let the law of force be the inverse n^{th} power of the distance or $r\phi r = \frac{1}{r^n}$;

$$\therefore \frac{\phi' r}{r} = -\frac{n+1}{r^{n+3}},$$

$$v^2 = -\frac{P}{2}\sigma\left\{\frac{1}{r^{n+1}} - \frac{(n+1)\delta y^2}{r^{n+3}}\right\}(\delta x^2 - \frac{\pi^2 \delta x^4}{\lambda^2} + \ldots\ldots),$$

as far only as the first half-wave.

If the motion be not in a vacuum, the above results are true provided that the material particles are at rest.

161. Let ϵ be the distance between two consecutive particles, $\delta x = \xi\epsilon$, $\delta y = \eta\epsilon$, &c.

$$\therefore\ v^2 = -\frac{P}{2\epsilon^{n-1}}\sigma\left\{\frac{1}{(\xi^2+\eta^2+\zeta^2)^{\frac{n+1}{2}}}\right.$$

$$\left.-\frac{(n+1)\,\eta^2}{(\xi^2+\eta^2+\zeta^2)^{n+3}}\right\}(\xi^2-\frac{\pi^2}{\lambda^2}\xi^4\epsilon^2+\&c.).$$

Now by reference to the case of light which we suppose an analogous one, we learn that when the velocity is greatest, the variation of velocity due to that in the value of λ is least: hence we should expect that for a few values of the numbers ξ, &c. the same hypothesis as to the value of ϵ which causes $\frac{1}{\epsilon^{n-1}}$ to increase, would cause $\frac{1}{\epsilon^{n-1}}\cdot\epsilon^2$ to diminish: this will be satisfied by making $(n-1)$ positive, and $(n-3)$ negative, or which amounts to the same thing, by making $n = 2$.

The law of the inverse square of the distance appears then to satisfy the requisite conditions. It is true we have failed to account for one circumstance, the explanation of which appears indispensible to the theory, viz. that in a vacuum there shall be an absolute freedom from variation of velocity due to a variation in the length of the wave.

This difficulty, we conceive, arises to a certain extent, from the imperfection of our acquaintance with the mode in which the particles may be arranged. One thing is certain, that when they are very close together, they act as a mass of particles in contact. Taking it for granted that such is the case, we may remove the difficulty thus.

162. Let D be the density of the particles in vacuo, then the equation of motion parallel to y becomes

$$\frac{d^2\beta}{dt^2} = -k^2n^2D.\beta.$$

But the mean density of the particles in the interior of a medium will be different from this, call it D'. In addition, a particle of matter will occupy a considerable space, which, if it were not there, would be filled with the particles of the medium.

Let Q be the repulsion at the distance unity of the mass of particles which would occupy the space of a material particle: then within the medium, we shall obtain as in (157),

$$\frac{d^2\beta}{dt^2} = -k^2n^2D'\beta - 2Q\beta\Sigma(\phi R + \frac{\phi' R}{R}\Delta y^2)\sin^2\frac{k\Delta x}{2}$$

$$- M\beta\Sigma(\phi R + \frac{\phi' R}{R}\Delta y^2) + M\beta'\Sigma(\phi R + \frac{\phi R}{R}\Delta y^2)\cos k\Delta x,$$

it being evident that $K = k$;

hence putting $\frac{1}{R^{n+1}}$ for ϕR,

$$\frac{d^2\beta}{dt^2} = -k^2n^2D'\beta - \Sigma\left(\frac{1}{R^{n+1}} - \frac{n+1}{R^{n+3}}\Delta y^2\right)$$

$$\times\left\{(2Q\beta + 2M\beta')\sin^2\frac{K\Delta x}{2} + M\beta - M\beta'\right\}.$$

Now if we suppose the law of the inverse square of the distance, the conditions (159) give

$$\because n = N,\ \Sigma F = \Sigma\frac{\Delta x^2 + \Delta y^2 + \Delta z^2 - 3\Delta x^2}{R^5}$$

$$= 0 \text{ in a medium of symmetry}$$

$$-\left(n^2 + 2\sigma f\sin^2\frac{k\delta x}{2}\right)a = -2\Sigma F\sin^2\frac{k\Delta x}{2}\cdot a'$$

$$-\left(n^2 + 2\Sigma F\sin^2\frac{k\Delta x}{2}\right)a' = -2\sigma f\sin^2\frac{k\delta x}{2}\cdot a;$$

and by addition $\alpha + \alpha' = 0$; $\therefore\ \alpha' = -\alpha$.

Similarly, $\beta' = -\beta$;

$$\therefore\ \frac{d^2\beta}{dt^2} = -k^2n^2D'\beta - 2\beta(Q - M)\Sigma F \sin^2\frac{k\Delta x}{2};$$

$$v^2 = n^2D' + 2(Q - M)\Sigma F . \frac{\sin^2\frac{k\Delta x}{2}}{k^2}$$

$$= \frac{D'}{D}V^2 + 2(Q - M)\Sigma F . \frac{\sin^2\frac{k\Delta x}{2}}{k^2};$$

V being the velocity in vacuo.

We shall not stop to discuss this formula, as the subject is too uncertain to allow us to pursue it into detail.

163. To find the values of n^2 and N^2.

The equations (157) are now reduced to

$$\frac{d^2\alpha}{dt^2} = 2P\alpha\sigma\left(\frac{1}{r^3} - \frac{3\delta x^2}{r^5}\right)\sin^2\frac{k\delta x}{2}$$

$$+ 2M\alpha\Sigma\left(\frac{1}{R^3} - \frac{3\Delta x^2}{R^5}\right)\sin^2\frac{k\Delta x}{2};$$

$$= T_x\alpha + U_x\alpha \text{ suppose};$$

$$\frac{d^2\beta}{dt^2} = 2P\beta\sigma\left(\frac{1}{r^3} - \frac{3\delta y^2}{r^5}\right)\sin^2\frac{k\delta x}{2}$$

$$+ 2M\beta\Sigma\left(\frac{1}{R^3} - \frac{3\Delta x^2}{R^5}\right)\sin^2\frac{k\Delta x}{2};$$

$$= T_y\beta + U_y\beta;$$

$$\frac{d^2\alpha'}{dt^2} = T_x\alpha' + U_x\alpha',$$

$$\frac{d^2\beta'}{dt^2} = T_y\beta' + U_y\beta'.$$

But by addition it is evident that

$$T_x + T_y + T_z = 0, \quad U_x + U_y + U_z = 0:$$

also it is clear that $T_y = T_z$, $U_y = U_z$;

$$\therefore\ T_x = -2T_y, \quad U_x = -2U_y;$$

or if $\dfrac{d^2\alpha}{dt^2} = 2n^2\alpha,$

$$\frac{d^2\beta}{dt^2} = -n^2\beta,$$

$$\frac{d^2\gamma}{dt^2} = -n^2\gamma,$$

$$\frac{d^2\alpha'}{dt^2} = +2n^2\alpha',$$

$$\frac{d^2\beta'}{dt^2} = -n^2\beta',$$

$$\frac{d^2\gamma'}{dt^2} = -n^2\gamma'.$$

164. The solutions of these equations are respectively

$$\alpha = fe^{n\sqrt{2}\cdot t} + ge^{-n\sqrt{2}\cdot t},$$

$$\beta = b\cos(nt - kx);$$

&c. = &c.

But we assumed the solution of the first equation to be

$$\alpha = a \cos (nt - kx).$$

It is necessary therefore to alter that assumption in such a manner as still to satisfy the values of δa, Δa found in (157).

Let

$$\alpha = a (e^{n\sqrt{2}\cdot t} - e^{-n\sqrt{2}\cdot t}) \cos (kx + c) = a' \cos (kx + c);$$

then δa will by the same operation as in (157) give the same results; and

$$\Delta \alpha = A' \cos (kx + c + k\Delta x) - \alpha$$

$$= a' \cos k\Delta x + \frac{da'}{dx} \cdot \sin \frac{k\Delta x}{R} - \alpha;$$

which is also the same as before.

The above therefore is a solution of the equation subject to all its assumptions.

165. But we must now make an important alteration in the form of δa.

It is evident that if the system of particles have a motion of *translation*, the force will no longer depend on α itself, but on its variation.

Hence instead of α put ψt, and instead of $g \cos (kx + c)$ put $-\frac{1}{k^2} \frac{d^2\alpha}{dx^2}$, and there results

$$\delta \alpha = + \frac{2}{k^2} \cdot \sin^2 \frac{k\delta x}{2} \frac{d^2\alpha}{dx^2} + \frac{d\alpha}{dx} \frac{\sin k\delta x}{k};$$

by the substitution of which in (157) we get

$$\frac{d^2a}{dt^2} = -\frac{1}{k^2}T_x\frac{d^2a}{dx^2} - \frac{1}{k^2}U_x\frac{d^2a}{dx^2} = -\frac{2n^2}{k^2}\cdot\frac{d^2a}{dx^2} \text{ for } \frac{d^2a}{dx^2} = -\frac{d^2a'}{dx^2};$$

the solution of which is

$$a = Vt + a(e^{n\sqrt{2}t} - e^{-n\sqrt{2}t})\cos(kx + c).$$

166. To interpret these results.

The form under which they appear manifestly shews that a motion of vibration is not possible at the same time in the direction of transmission and transverse to that direction; from which we may conclude, that in the case of heat, the vibrations are altogether transversal.

From the last equation we learn that a motion of translation may take place in such a manner that a particle is alternately accelerated and retarded by means of its vibrations.

167. But we have supposed the medium one of symmetry; we must now make the hypothesis that the symmetry only holds with respect to the axis of x, there being more caloric in advance than behind, or *vice versâ*.

Let us examine the equation for motion parallel to y; we find

$$\sigma\{\phi r + \frac{\phi' r}{r}(\delta x\delta a + \delta y\delta\beta + \delta z\delta\gamma)\}(\delta y + \delta\beta)$$

$$= \sigma\{\phi r\delta y + \phi r\delta\beta + \frac{\phi' r}{r}(\delta x\delta y\delta a + \delta y^2\delta\beta + \delta y\delta z\delta\gamma)\}$$

$$= \sigma(\phi r + \frac{\phi' r}{r}\delta y^2)\delta\beta.$$

Suppose the same form of $\delta\beta$ to hold; then

$$\frac{d^2\beta}{dt^2} = +m^2\frac{d^2\beta}{dx^2} - s\frac{d\beta}{dx},$$

for $\sigma f \sin k\delta x$ is not equal to zero.

Let $\beta = be^{ht}\cos(nt - kx)$;

$$\therefore \frac{d\beta}{dt} = -bne^{ht}\sin(nt - kx) + bhe^{ht}\cos(nt - kx),$$

$$\frac{d^2\beta}{dt^2} = -n^2\beta - 2bhne^{ht}\sin(nt - kx) + h^2\beta,$$

$$\frac{d^2\beta}{dx^2} = -k^2\beta,$$

$$\frac{d\beta}{dx} = kbe^{ht}\sin(nt - kx);$$

$$\therefore (n^2 - h^2) = k^2m^2,$$

$$2hn = sk,$$

$$h = \frac{sk}{2n},$$

$$n^2 - \frac{s^2k^2}{4n^2} = m^2k^2.$$

These results do not introduce any new observations; the vibration will still exist in a certain sense, but the motion will vary with the time explicitly, as it obviously ought to do.

168. If now we resume the examination of the equations in α, we find that there is a constant term in the expression for the force, which term we shall mention as probably introducing a very important distinction between the cases of heat and light. It is not improbable that the effect of this term is in the former subject a most important one, giving rise to the principal part of the whole action, whilst in the latter it is a subsidiary one, producing little or no direct variation in the general result. We have already stated the possibility of this in (122), but experiments are too scanty on this head to enable us to speak with any degree of certainty.

Now

$$\sigma(\delta x+\delta a)\phi(r+\rho)=\sigma\frac{\delta x+\delta a}{\{(\delta x+\delta a)^2+(\delta y+\delta\beta)^2+(\delta z+\delta\gamma)^2\}^{\frac{3}{2}}}$$

$$=\sigma\frac{\delta x+\delta a}{\{r^2+2(\delta x\delta a+\delta y\delta\beta+\delta z\delta\gamma)+\delta a^2+\delta\beta^2+\delta\gamma^2\}^{\frac{3}{2}}}$$

$$=\sigma\frac{\delta x+\delta a}{r^3}\left\{1-\frac{3}{2}\frac{2(\delta x\delta a+\delta y\delta\beta+\delta z\delta\gamma)+\delta a^2+\delta\beta^2+\delta\gamma^2}{r^2}\right.$$

$$\left.+\frac{15}{8}\frac{4(\delta x\delta a+\delta y\delta\beta+\delta z\delta\gamma)^2}{r^4}\right\}$$

$$=\sigma\frac{\delta x+\delta a}{r^3}-\frac{3}{2}\sigma\frac{\delta x^2\delta a+\delta x\,\delta a^2+\delta x(\delta a^2+\delta\beta^2+\delta\gamma^2)}{r^5}$$

$$+\frac{15}{2}\sigma\delta x\frac{\delta x^2\delta a^2+\delta y^2\delta\beta^2+\delta z^2\delta\gamma^2}{r^7}$$

$$=\sigma\frac{\delta x}{r^3}-\frac{3}{2}\sigma\frac{\delta x}{r^5}(\delta\beta^2+\delta\gamma^2)+\frac{15}{2}\sigma\frac{\delta x}{r^7}(\delta y^2\delta\beta^2+\delta z^2\delta\gamma^2)$$

$$=C+p(\delta\beta^2+\delta\gamma^2)+\&c.$$

$$=C+p(b^2+c^2)+\&c.$$

the terms which we omit being those already taken into account in the previous investigation.

169. In order to adapt this conclusion to the case before us, that of the transmission of heat through media, we must observe that such is the constitution of matter, that the density of the particles of caloric within a medium is greater than that without; but that owing to the attractions, repulsions and pressures of the different constituent parts, the whole is in a state of equilibrium, provided the temperature is uniform. Now the quantity C in the preceding result is the aggregate attraction on a particle, and is consequently zero in the case supposed. Nor can it be very considerable even when the temperature is not uniform except under the circumstances of heat passing from a hot

body into a cold one *directly*. When heat is emitted from a source, and in its progress towards a colder substance has to pass *across* the medium under consideration, the difference between the actual state without the medium and that within will not be considerable. In such circumstances C will have only a small value at the surface of the medium, whereas the conditions will not at all affect the value of $a^2 + b^2$, and consequently the amount of the force at the surface of the medium due to the vibration will in general vary as that due to the direct repulsion, and have the same sign. This will not, however, be always the case: we know that in light the nature of the medium is such that under many circumstances the magnitude of a is very different from that of b, so that if the original vibrations be not equal in all directions, the quantity of the transmitted action will depend on the relative positions of the planes of incidence and vibration. We are then at liberty to adopt the same hypothesis for heat, and thus all the characteristics of polarization, &c. will be found to be common to the two.

170. The characteristic of polarization is that the vibrations in two directions transverse to that of transmission are transmitted with different velocities. The mode in which this can be conceived belongs to our statical investigation, which we shall adduce presently. We shall assume at present that the distribution of caloric within a uni-axal crystal is such that the distance of consecutive particles in a direction parallel to the axis is less than the corresponding distance in a plane perpendicular to the axis, or *vice versâ*. In other words, eight particles may be supposed to be placed in the corners of a parallelopiped whose base is a square. Within a biaxal crystal the analogous representation will be a parallelopiped whose edges are all of unequal length.

171. Whether this be considered merely as a representation to the mind or as a direct supposition is not of much

importance. We will not pretend to attach too much weight to it. It is however very simple of application.

Call the distance between two consecutive particles, estimated as we have supposed, the *linear distance.* Then if the motion be a motion of vibration in the direction of the axis of x_1, it is evident that the pressure on a plane of given area parallel to y_1 z_1, will vary as the density of the particles in that area, that is, (since a given volume always contains the same number of particles) directly as the linear distance parallel to x_1. Now the elastic force parallel to x_1 in a given volume varies as that on a given plane directly, and the number of planes so contained inversely. Also the number of planes varies inversely as the linear distance; consequently the elastic force parallel to x_1 varies as the square of the *linear distance* along that axis.

In order to calculate this expression, let the axes of x, y, z be the axes of symmetry. Take a spherical portion of the mass, and whilst the particles retain their distances parallel to x, let those parallel to y and z be drawn out or the reverse until the density is the same in all directions and the mass uniform. The sphere, therefore, will become an ellipsoid, having its axes proportional to the reciprocal of the linear distances in their directions respectively. It follows, therefore, that the elastic force along the axis of x, is proportional to the inverse square of the radius of the ellipsoid which is parallel to the direction of x_1.

Let $a^2x^2 + b^2y^2 + c^2z^2 = 1$ be the equation to the ellipsoid, and let the axis of x_1 make the angles θ, ϕ, ψ with x, y, z respectively.

Let r be the length of the radius required,

$$\text{then } r\cos\theta = x, \text{ \&c.}$$

$$\therefore (a^2\cos^2\theta + b^2\cos^2\phi + c^2\cos^2\psi) = \frac{1}{r^2}.$$

and if a^2 be the measure of the elastic force parallel to x, $a^2\cos^2\theta + b^2\cos^2\phi + c^2\cos^2\psi$ will represent the elastic force in any direction making θ, ϕ, ψ with x, y, z.

172. Let the plane of vibration cut the ellipse in the plane x_1y_1, the axis of y_1 making the angles θ, ϕ, ψ with the axes of x, y, z: then it is necessary for the continuance of a vibration that the direction of motion should be symmetrically situated with respect to the particles; or that the plane through the direction of transmission and that of vibration should be a plane of symmetry in the medium. It follows, therefore, that the axes of x_1, y_1 must be the principal axes of the elliptic section by the plane x_1y_1.

173. We do not purpose to proceed further in this investigation. We have said enough to make it obvious that rays will in general proceed with different velocities within a medium, depending on the direction of their vibration, and that consequently a single ray will be divided into two; so that on emerging from a crystal the nature of the vibrations may be very considerably altered.

The phenomena of depolarization, circular polarization and others have been already discussed on the hypothesis which we have just established, that vibrations within a crystal are transmitted with velocities dependent on the direction of vibration. We will therefore dismiss this subject referring the reader to Airy's Mathematical Tracts, and the seventh volume of the Memoirs of the Institute for further information.

174. We come now to the consideration of the statical part of our problem. It is far more beset with difficulties than that just discussed, both from its nature, and from the degree of uncertainty that rests upon the facts to which we must have recourse for its establishment. The following sketch has no pretensions to more than an attempt at classifying the phenomena under one hypothesis, exacting from

that hypothesis a rigorous determination of the circumstances which it is supposed to embrace, only in those cases in which we possess sufficient knowledge to enable us to apply the investigation with certainty; so as to pronounce judgment should the hypothesis be insufficient.

175. The body whose constitution we shall imagine ourselves discussing is a simple gas consisting of a series of material particles of the same kind, each of which is surrounded by its atmosphere of caloric. The particles of gas will be conceived as very much larger than those of caloric, so that each of them may be accompanied by a considerable number of particles of the latter kind arranged about it in such a manner that their density is greater as they lie nearer to its surface. It is supposed that two molecules of matter as well as two molecules of caloric mutually repel each other, but that a molecule of gas is attracted by a molecule of caloric. This hypothesis (which is the common one) leads to the conclusion that the particles of caloric next a particle of gas are in actual contact with its surface, and consequently *press* against it. Their action will therefore be of a twofold and opposite character; by their attraction tending to pull it, by their contact (being repelled by the particles next them) tending to push it. We desire to direct especial attention to this hypothesis as it enables us to remove a difficulty so considerable, as to have induced Laplace and others after him to reject the simple and obvious law of attraction and repulsion according to the inverse square of the distance, and to substitute in its place a law difficult to be applied, and impossible (we think) to be represented to the mind. Laplace's solution, however, is just such an one as *corresponds* to the state of things, requiring that the action of contiguous particles should be of a different nature from that of distant ones; but we object altogether to the introduction of a mode of action with which the mind is not familiar, and which serves no other purpose than to explain the particular point for which it is invented, even if it does so much.

We suppose that gases differ from one another either by the size of their particles or the attractive power which a particle exerts on caloric, or by both combined; it will be of little consequence. The gas is conceived to be confined in an envelope, and the pressure exerted on this envelope is the pressure of the gas.

We proceed to apply the hypothesis to the solution of the following problems.

176. Prob. The pressure of a gas, under a constant temperature varies as its density.

The definition which we shall give to constant temperature is this, that the state of the caloric is such as to give a constant pressure against a particle of gas.

For the solution of the problem before us, we must observe that we are not concerned with the actual force on any particle, for if that force be the same in all directions, its absolute effect will be zero. All we have to consider is the excess of pressure exerted on a particle of gas by its atmosphere of caloric above that which would be exerted on it, if the whole system were of one uniform density. Suppose then the number of particles in a given space to be doubled, the force on each remaining unchanged, it is clear that the force on the surrounding envelope will be doubled; for two particles may be supposed to fill the place before occupied by one. From this reasoning it is obvious, that when the temperature remains unchanged, the pressure will vary as the density.

177. Prob. When two gases of the same temperature are mixed, the whole pressure is the sum of the pressure due to each individually.

By the former proposition it appears that the action on any particle of gas is independent of the presence of contiguous particles, provided only the pressure of the caloric on a particle be not altered. Hence the truth of this pro-

position is evident. We have every reason too for the adoption of the hypothesis in the form in which we conceive it. For the caloric within the substance is in equilibrium with that without, both in the simple and the mixed gas; hence the pressure of the caloric at the confines of the medium will be the same in both cases, and by its action equilibrium is preserved with that within, consequently its pressure within the medium ought not to differ from that without.

178. PROB. When two gases of different kinds are suffered to communicate, they mix amongst each other so as to attain finally the same density through the whole space.

It is obvious that unless the gases are precisely similar in their constitution and of equal density, they must mix amongst each other. For a particle of the gas A presenting itself at the space between two particles of B, sustains from them no action except that which arises from the caloric which surrounds them. But this caloric attaches itself to a particle of B, by virtue of the attractive force which that particle exerts on it. Now as the particle of A has also an attractive force for it, an interchange will take place until the caloric which is attached to each of the particles acquires the same pressure, and the particles are thus left to penetrate betwixt each other. This process will go on until the mixture is complete, except in the case where the gases are of the same kind, when it will go on only until the two portions have acquired the same density.

179. PROB. A gas expands by the same fraction of its volume at 0^0, for every degree of temperature.

Let t be the temperature corresponding to the volume V', then if V be the volume at 0^0,

$$\text{pressure at } 0^0 = c,$$

$$\ldots\ldots\ldots\ldots\ t^0 = \frac{c'V}{V'},$$

where c' is the value of the pressure of the caloric on a particle: it is evident from our definition of temperature that $c' = c + \beta t$; therefore equating the pressures

$$c = \frac{c'V}{V'} = (c + \beta t)\frac{V}{V'},$$

$$V' = V\left(1 + \frac{\beta}{c}t\right).$$

Cor. The quantity c expresses the pressure of the caloric when the temperature $= 0$, and therefore varies as the density of the caloric. Similarly, $c + \beta t$ measures the density of the caloric at t^0; if therefore the density of the caloric vary as the attractive force of the particles of gas $\frac{\beta}{c}$ will be the same for all gases.

180. To find the state of equilibrium of the particles of gas and those of caloric.

In order that a particle of caloric may be in equilibrium, it is necessary that the attraction of the material particles should diminish its tendency to move in a given direction, just as much as the repulsion of the caloric particles increases that tendency. In other words, if caloric particles alone existed extending to infinity in all directions, the action on the particle would be zero; if again material particles were added to caloric ones so as to extend also to infinity, the action on a particle similarly situated with respect to all its adjacent particles would still be zero.

When neither of these conditions holds, a particle will suffer variation of force arising from two circumstances: 1st, its position in the whole mass, depending on the proximity it has to the boundary: 2nd, its position with respect to the adjacent particles. We shall discuss these circumstances in the following problems.

181. PROB. Two series of particles are contained in a spherical envelope, which is surrounded by an infinite mass of particles of caloric, find the conditions of their distribution that there may be equilibrium.

Let a be the radius of the envelope,

r the distance of any particle of the caloric from its center,

D the density of the caloric at that point,

D' that of the gas; $\mu D'$ its attraction on caloric;

then the attraction of a shell of the gas on a particle of caloric at the surface is

$$\frac{4\pi r^2 \delta r}{a^2} \mu D',$$

and the repulsion of the caloric will be a quantity of the same kind, but as we shall have to consider the difference between the actual density of the caloric and that of the external medium, or rather of a mean external medium, we must take account of the caloric displaced by the gas, the quantity of which will be $cD'D$: hence the repulsion of caloric

$$= \int_0^a \frac{4\pi r^2 dr}{a^2} (1 - cD') D;$$

therefore the whole force of repulsion on a particle of caloric

$$= \int_0^a \frac{4\pi r^2 dr}{a^2} \{(1 - cD') . D - \mu D'\},$$

but if D_1 be the mean density mentioned above, its repulsion is $\int_a^0 4\pi r^2 dr . D_1$; and since there is equilibrium, we must have

$$D(1 - cD') - \mu D' = D_1,$$

$$D = \frac{\mu D' + D_1}{1 - cD'};$$

$$D - D_1 = \frac{\mu D' + cD'D_1}{1 - cD'}$$

$$= \frac{(\mu + cD_1)\, D'}{1 - cD'}$$

$$= D' (\mu + cD_1 + \mu cD') \text{ nearly,}$$

$$= \mu D' \text{ nearly.}$$

It appears from this result that the excess of density of the particles of caloric amongst those of gas corresponding to a given temperature is proportional to the attractive force of the particles of gas.

Cor. By the corollary to (179) it appears that all gases will expand equably by equal additions of temperature, provided the action of the caloric varies as that of the particles of gas or as μ. Now the action will be a simple function of $D - D_1$ depending on the law of distribution to be hereafter determined, hence we conclude that the corollary referred to holds true.

182. Prob. When a mass of gas is compressed, to find the quantity of caloric evolved.

Suppose the temperature of the gas to be the same after the compression as before.

Let V, D be the volume and density of the caloric before compression,

V', Δ afterwards,

D_1 the mean density as in Art. (181),

D' the density of the material particles before compression,

Δ' afterwards;

$$\text{then } \Delta' = \frac{D'V}{V'}.$$

But by Art. (181) $D - D_1 = \mu D'$,

$$\Delta - D_1 = \mu \Delta'$$

$$= \mu \frac{V}{V'} D'.$$

Now the quantity of caloric evolved is

$$VD - V'\Delta = VD_1 + V\mu D' - V'D_1 - V\mu D'$$

$$= (V - V')\, D_1;$$

from which it appears that the total quantity of caloric evolved by compression is independent of μ, that is, of the *nature* of the gas, and M. Dulong has proved experimentally that by compression all gases give out the same quantity of heat.

Cor. To find the temperature of the compressed gas.

Let c be the specific heat of the gas, under a constant volume; then if t be the original or final temperature; t' that at the instant of compression,

$$(V - V')\, D_1 = c\,(t' - t)\, V';$$

$$\therefore t' - t) = \frac{(V - V')\, D_1}{c\, V'} = \left(\frac{V}{V'} - 1\right) \frac{D_1}{c};$$

i. e. the change of temperature is inversely as the specific heat, which agrees with M. Dulong's experiments.

183. Prob. To find the state of equilibrium of a particle situated any where within the medium as affected by the proximity of the particles of gas.

By the preceding proposition a particle of caloric will be unaffected by the action of the particles at a distance

from it so far as they tend to give it motion, the attraction or repulsion in one direction being exactly counterbalanced by that in the opposite direction. This is however not to be effected without a variation of density of the caloric particles, depending on their position with respect to any given particle of gas.

We may in this investigation conceive the distribution to be symmetrical with respect to a particle of gas; and consequently may conceive all the caloric which is aggregated about any material particle to be collected at the center of that particle.

Let the center of a particle of gas whose radius is l be taken as the origin; then, if a be the distance of the particle of caloric from the origin, $D.m$ the excess above the mean of the repulsion from the caloric contained in any spherical stratum m:

the repulsion of the extra caloric surrounding this particle

$$= \int_l^a \frac{4\pi r^2}{a^2} dr \,.\, D;$$

and the attraction of the molecule $= \frac{\mu}{a^2}$.

Hence if we omit the consideration of the other particles which may be done when that under consideration lies near the surface of the material particle, it is obvious that these expressions must be equal, or

$$\frac{\mu}{a^2} = \frac{4\pi}{a^2} \int_l^a r^2 D dr;$$

$\therefore \int_l^a D r^2 dr$ is independent of a, which is the condition in this case.

184. To find a value of D which satisfies the conditions.

We shall not attempt to seek for a solution under a general form, but content ourselves with selecting one which obviously satisfies the conditions, which are that D have a value such that $\int_l^a r^2 D dr$ shall not contain a.

$$\text{By Art. (68)} \begin{cases} \sin l \cos a + \frac{1}{2} \sin 2l \cos 2a + \ldots\ldots \\ + \operatorname{vers} l \sin a + \frac{1}{2} \operatorname{vers} 2l \sin 2a + \ldots\ldots \end{cases}$$

is independent of a so long as its value lies between l and 2π.

This solution then completely satisfies the conditions, and we obtain as the value of D,

$$D = Ar^3 \begin{cases} - \operatorname{vers} l \cos r - \operatorname{vers} 2l \cos 2r - \ldots\ldots \\ + \sin l \sin r + \sin 2l \sin 2r + \ldots\ldots \end{cases}$$

$$= Ar^3 \{ - \cos r - \cos 2r - \ldots\ldots$$
$$+ \cos (r - l) + \cos 2(r - l) - \ldots\ldots \}$$

185. There is an essential difference in the constitution of the gases and that of steam; the former exerting an attraction on heat will as we have seen exert a greater pressure when the quantity of heat contained amongst their particles is increased; the latter not exerting any extra pressure by the increase of heat, must consequently be supposed neutral with respect to the caloric. The results of the gases will consequently apply to steam by putting $D - D_1 = 0$, and therefore supposing the pressure constant. We shall not stop to discuss this hypothesis.

186. PROB. To find the state of the equilibrium of a particle of the gas.

We have already stated that we are to conceive an arrangement of the mass of such a nature, that as far as attractions and repulsions are concerned, an increase of density will in general have no sensible effect on the force acting on a particle. Such being the case it is evident that an increase of heat above the mean will cause by its *pressure* an

increased tendency in the particle to recede from its position. Hence the general conclusion is this; that an increase of heat within bodies above the state without tends to cause their particles to fly off from one another, and thus will evaporation, &c. be accounted for.

187. Cor. It appears that fluids differ from gases in this respect, that the actual force exerted on their particles is one of *attraction*, i. e. that the quantity of caloric (or in some cases its arrangement) within them is such as to press against them less than the external pressure corresponding to a given temperature. We do not mean the external pressure of the air in this case, but that of the caloric which we have supposed to pervade all space.

We must be careful to observe that the equations $D - D_1 = \mu D'$ and the like, do not impose on us the necessity of supposing that the pressure on the particles is of necessity greatest within the medium in all cases in which the force of the matter is attractive for that of caloric, nor does it follow that an increase of density amongst the material particles causes an increase of pressure; for the same causes which operate to increase the caloric within, will also operate to increase that without the medium in the neighbourhood of the particles.

The tendency of *contiguous* particles is to withdraw the internal caloric from the surface of the terminal one, and to accumulate the external caloric about it, so that the closer the material particles are, the more will they tend to induce a state of equilibrium.

188. We proceed now to the more complex forms of bodies, the solid and crystalline, and here we must beg some latitude of hypothesis, as the subject is rather uncertain.

We supposed that gases of different kinds are composed of particles in such combination with caloric, that they do not exert on each other any sensible action when in their

gaseous state; we then conceived the quantity of caloric diminished until the particles assumed a state in which they were pressed together, which is the fluid state; we are now to consider another transition, viz. to the solid state. In order to fix the ideas we will take the case of water in the act of freezing, and suppose it composed of alternate particles of oxygen and hydrogen forming an uniform mass. Let us conceive the caloric continually abstracted until a particle of oxygen is thrust by the pressure against it on one side next the air, until it actually touches the particle of hydrogen immediately within it. At that instant a complete change takes place in the conditions, arising from the circumstance that the portions of surface in contact are no longer pressed out by the caloric. It is clear too that a change of volume will take place for this arrangement will require a totally new distribution of the caloric, not very much differing from that in a simple system composed of particles of one kind only. It may however be observed, that the pressure of the gas on the compound particle will be the same as on a simple one, provided there be just so much caloric removed as to allow the density to remain the same as before on the sides of the particles not in contact: this requires that nearly half the heat should be given out by freezing under the circumstances in question, and completely explains the subject of latent heat.

189. The mode of explanation which we have adopted, viz., that the change to a state of fluidity from the gaseous is due to such an arrangement of the particles that their equilibrium is no longer one which requires to be retained by the external pressure, and that the subsequent change to a solid form is brought about by the union of particles of different kinds into a single system, is one which we do not insist on as being more probable than any other; we merely adopt it as one of many that might be devised. It is highly probable that the union of particles either of the same or of different kinds takes place at liquefaction, which particles by the abstraction, &c. of caloric commence to exert

sensible attractions on each other so as to influence the position of those next them, and thus to induce an arrangement in order to break which it is necessary to introduce a considerable quantity of caloric. Thus if a fluid consists of pairs of particles of different kinds united, the nature of a fluid might be that these pairs were at such distances from each other compared with the quantity of caloric amongst them, that the line joining the centers of any pair forming one combination would admit of any direction with respect to lines similarly drawn in other combinations, the small variations of force introduced by a change of position being balanced by small variations in the surrounding caloric. But when a quantity of caloric had been abstracted, some state would be arrived at for which this indifference of position was no longer possible, and consequently a slight motion of the particles would result which would throw them into positions dependent mutually on each other, as for instance all the axes may become parallel to one another, or alternately parallel and perpendicular. We conceive this is the more probable hypothesis, and it receives strength from the consideration of the constitution of crystals which follows.

190. We do not intend in the present work to enter at any length into the subject of crystallization, it will suffice that we make a few remarks as to the mode of arrangement. If we suppose particles endued with different attractive forces for caloric to be united two and two, the combined particles, which act on each other with an intensity different from that with which they act respectively on particles of their own kind, must be arranged in a symmetrical order. Hence the distribution of caloric is such that a *line* of given length drawn in one direction will not in general encounter so many particles as the same line drawn in another direction; in fact, the mean arrangement of the particles is analogous to that of points in the corners of a parallelopiped with a square base, and a height not equal to a side of the square. Indeed every combination

of material particles in this mass may be considered as a spheroid, and the consequent aggregation of caloric extending to a certain distance from it, will also have a spheroidal arrangement.

Again, imagine any number of particles of different kinds mixed together with caloric about them; the same difference in their attractions and repulsions which was before mentioned will produce the same effect of arranging each set symmetrically, so that the medium will still be one of unequal symmetry. Now whatever be the number of particles united, such is the nature of forces that the system cannot remain in equilibrium, or at least in stable equilibrium, except the different *sets* of particles have arrangements analogous to each other, and so distributed that there is a regular recurrence of the same form; and therefore any one combination, if not altogether independent of the others will certainly only be affected by a small number of those next it.

Now whatever be the distribution it is clear that any one complete system comprising all the variations will admit of three lines of symmetry within it analogous to the three principal axes of a solid body. The system of caloric will therefore possess all the properties requisite for the explanation of the phenomena of biaxal crystals, for the arrangement is analogous to the ellipsoidal.

191. Prob. To find the effect of a change of temperature in a uniaxal crystal.

Let y be the distance of a particle of caloric from the center of the material particle next it, and a the radius of either of the two particles (arranged as in the last proposition) the force on the particle of caloric will vary as $\frac{1}{y^2} + \frac{\mu}{(y+a)^2}$ if it lie in the line joining their centers, and as

$$\frac{\sqrt{y^2-a^2}}{y^3} + \frac{\mu\sqrt{y^2-a^2}}{y^3}, \text{ or as } \frac{(1+\mu)\sqrt{y^2-a^2}}{y^3},$$

if it lie in a direction at right angles to that line, from its middle point. Call these attraction P and Q respectively. When $y = a$ it is clear that the latter expression is zero, as we have considered it to be; but as y increases the latter expression increases more rapidly than the former, and at a certain distance exceeds it in value; as for instance if $\mu = 1$, when $y = 10a$, the latter expression is the greater.

Let D be the density of the caloric, D_1 the mean density at the same point, then $D - D_1$ varies as force of attraction, consequently the accumulation will be greatest at the points of greatest attraction.

Suppose the temperature increased, then since $P - Q$ remains constant, the cumulation of caloric will not be increased: there will it is true be a slight change of position amongst the particles of caloric owing to the actual expansion of the crystal; but we omit the consideration of this circumstance as it does not affect the present discussion. It follows clearly then that *in this arrangement* an increase of temperature will tend to render the system symmetrical in all directions by diminishing the eccentricity or unequal cumulation of the particles. This agrees exactly with the experiments of Mitscherlich. But yet further, if another combination lie in the direction of the line which is the major axis of the former, their mutual distance depends not on the actual density of the caloric at their ends, but on the combined effect of its pressure with the attraction of the caloric not immediately in contact with them. Now the new caloric introduced has been so distributed that a greater quantity in some cases will be attached to the sides than to the ends of the particles in contact; measuring quantity by the actual density; it is clear then that the attractive force in the direction of the axis has been increased to a greater extent than the pressure in the same direction, and that a consequent *contraction* in that direction will be the result. In such a case however there must be an expansion in other directions. The converse case is probably the one in Iceland spar.

192. PROB. To explain the analogous properties of biaxal crystals.

If we suppose a biaxal crystal to be composed of an arrangement of particles three and three, it will readily follow that the distribution of the accompanying caloric is in the form of an ellipsoid. But by the last proposition it appears that the excess of density is variable at different distances, being greatest at the extremity of one axis of an elliptic section, in the immediate neighbourhood of the particles, and least in the same direction at a considerable distance. The accumulation will therefore depend to a certain degree on the extent to which the caloric spreads from the particle. Now as the temperature is increased that extent is enlarged, hence there will be actually an increase of total accumulation at the extremity of the perpendicular from the center of the line joining two particles. As the temperature continually increases, this additional accumulation increases until the density in the section we are considering is uniform. After this the greater density is in the other direction, and the optic axes of the crystal lie in a plane perpendicular to that in which they formerly lay.

193. To find the variation in the density of the caloric caused by the action of the contiguous particles.

In Art. 184 we calculated the force at any distance, in order that the particles of caloric may be in equilibrium. We propose in the following Article to complete the subject by determining the law of density on the same supposition, and shewing the possibility of such a state of equilibrium.

Adopting a notation analogous to that in Art. 156, let us designate by x, y, z, the co-ordinates of a particle of caloric measured from the center of an atom of matter, by x', y', z', those of another particle, by X, Y, Z, those of a

material particle, and retain the same letters for the attractions and distances as in that Article.

$$\text{Let } V = P \iiint \frac{dx'\, dy'\, dz'\, D}{r};$$

calling the density of caloric D, then the repulsion of the mass of caloric on a particle P is $P.\frac{dV}{dx}$ in the direction of the axis of x;

$$\text{and if } U = \Sigma \frac{M}{R},$$

the attraction of the matter on the same particle is $P\frac{dU}{dx}$ in the same direction: hence the force by which the particle is pulled in the direction of x is

$$P\frac{dU}{dx} - P\frac{dV}{dx};$$

and the force by which the volume $\delta x\, \delta y\, \delta z$ is solicited is

$$D\, \delta x\, \delta y\, \delta z \left(P\frac{dU}{dx} - P\frac{dV}{dx} \right).$$

Let p be the pressure on an unit of the area $\delta y\, \delta z$ nearest the origin, then $p + \frac{dp}{dx}\delta x + \ldots$ is that on the plane furthest from it, and

$$\frac{dp}{dx} = PD\left(\frac{dU}{dx} - \frac{dV}{dx}\right).$$

Now if we omit the consideration of the variation of force on an individual particle compared with the variation

of density, the pressure on any surface will vary jointly as the number of pressing particles and the number pressed, that is, as the square of the density.

Let $p = \frac{1}{2} c D^2$;

$$\therefore \frac{dp}{dx} = cD \frac{dD}{dx};$$

$$\text{and } \therefore \frac{dD}{dx} = \frac{P}{c}\left(\frac{dU}{dx} - \frac{dV}{dx}\right);$$

$$\frac{d^2D}{dx^2} = \frac{P}{c}\left(\frac{d^2U}{dx^2} - \frac{d^2V}{dx^2}\right);$$

and similar expressions obtain in y and z,

$$\left.\begin{aligned} \text{but } \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} &= -4\pi D \\ \frac{d^2U}{dx^2} + \frac{d^2U}{dy^2} + \frac{d^2U}{dz^2} &= 0 \end{aligned}\right\}.$$

(See Pratt's Mechanical Philosophy, p. 155.)

$$\text{Hence } \frac{d^2D}{dx^2} + \frac{d^2D}{dy^2} + \frac{d^2D}{dz^2} = -\frac{P}{c}\left(\frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2}\right).$$

$$= \frac{4\pi P}{c} D$$

$$= \alpha^2 D;$$

the solution of which equation is

$$D = \Sigma . \frac{A e^{-\alpha R}}{R}.$$

The expression Σ is to be considered as applying to the different points at which a new order of periodicity occurs, and consequently will have reference to the centers of the different atoms of matter.

M. Mossotti makes D the difference between the density of the caloric and the mean density in space.

In order to determine the conditions of equilibrium of a particle of matter, we must observe that the forces which act on it are, the pressures and the attractions and repulsions. Now V is nearly of the same form as D, that is, it is a function of $\frac{1}{R}$ and $\frac{e^{-\alpha R}}{R}$, so that the equation of equilibrium assumes the shape

$$\Sigma\left(A\frac{e^{-\alpha R}}{R}-\frac{B}{R}\right)-C=0;$$

and the particles nearest the one under consideration will have the principal influence provided α is very large, which is no doubt the case; as the expression enters into that for the velocity of light (160). When therefore the particle is at the confines of a medium, it will depend on circumstances whether the pressure inwards combined with the additional attraction, shall overcome the repulsion of the adjacent particles or not.

When R is equal to the radius of a particle the corresponding term in the density is very large, but the *attraction* will depend not on the aggregation of caloric at the surface of the particle simply, but on the total aggregation between that surface and the middle point between two consecutive particles; which diminishes very rapidly on account of α. On the other hand, the *pressure* will depend entirely on the aggregation at the surface of the particle, and will therefore increase much more rapidly than the attraction.

Hence an excess of caloric will tend to destroy the equilibrium; a conclusion to which our hypothesis could not by any means lead without the consideration of direct pressure.

Since C remains constant, whilst A and B vary with the quantity of caloric, it is clear that when the caloric is much diminished, the same reasoning that would make the force on the bounding particle to act outwards by the excess of pressure due to a large supply of caloric would make it act inwards by the excess of attraction for a small one, and moreover, an increase of R by the removal of the particle to a little distance would scarcely affect the attraction, whilst by diminishing the pressure within, it would introduce a force whose tendency would be to cause the particle to return to its former position.

We have thus a complete view of the state of equilibrium of solid and gaseous bodies. The intermediate state of fluidity requires that the equation above should not perceptibly change by a variation in R, and consequently its first differential coefficient with respect to the distance between two particles must vanish.

It is not within the limits of the present work to discuss these results at length. One conclusion obviously follows, viz. that when a solid and fluid are in contact, the density of caloric about the particles of the fluid near the solid will be diminished in some proportion to the difference of the specific heat of the fluid and that of the solid (the latter however being estimated differently from the former). Whence it would appear that the altitudes to which fluids rise in Capillary tubes will vary as some simple function of their specific heat. We do not know how far this is borne out by experiment.

194. We shall now close the subject by recapitulating our theory as follows. Bodies are conceived to be made

up of repulsive molecules surrounded by atmospheres of particles attractive towards them, but repulsive to one another. The temperature of a body is estimated by the pressure of the attractive particles on those of matter with which they are in actual contact. Transmission is effected by means of vibrations of the particles of caloric, and the quantity of caloric transmitted varies as the intensity of vibration. The law of force is that of the inverse square of the distance.

THE END.

ERRATA.

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